Department of Mathematics OSU
Qualifying Examination
Fall 2012

PART I : Real Analysis

- Do any four problems, with solutions written in your blue examination book, and indicate on the selection sheet with your identification number those problems that you wish graded.

- Your solutions should contain all mathematical details. Please write them up as clearly as possible.

- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.

- On problems with multiple parts, individual parts may be weighted differently in grading.

- You have three hours to complete Part I.

- When you are done with the examination, return the examination blue book and selection sheet to the envelope. You will hand in all materials. If you use extra examination books, be sure to place your code number on them.

1. On the half open interval $[0, 1)$ define the k functions

$$f_{k,1}(x),\ f_{k,2}(x),\ f_{k,3}(x),\ \ldots,\ f_{k,k}(x),$$

for every integer $k \geq 1$ by $f_{k,i}(x) = 1$ for $x \in [\frac{i-1}{k}, \frac{i}{k})$ and by $f_{k,i}(x) = 0$ otherwise. Construct the sequence g_n by taking the $f_{k,i}$ with the standard lexicographic ordering on the indices (k, i):

$$g_1 = f_{1,1},\ g_2 = f_{2,1},\ g_3 = f_{2,2},\ g_4 = f_{3,1},\ g_5 = f_{3,2},\ g_6 = f_{3,3},\ g_7 = f_{4,1},\ \ldots\ .$$

(a) Show that g_n converges in measure to 0.

(b) Show that g_n is not pointwise convergent at any $x \in [0, 1)$.

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function such that

$$\lim_{x \to +\infty} x f'(x) = 0 \quad \text{and} \quad \lim_{n \to \infty} f(2^n) = 0.$$

Prove that $\lim\limits_{x \to +\infty} f(x) = 0$.

Hint: Let $g(y) = f\left(\dfrac{1}{y}\right)$ and show that $\lim\limits_{y \to 0+} y g'(y) = 0$.

3. Find two norms on the space $C([0, 1])$ of continuous real-valued functions on $[0, 1]$ such that the complement of the unit ball in one norm is dense in the unit ball of the other norm.

(exam continues on next page)

4. Given $R > 0$, and $f(x) \in L^1[0,R]$. Show that

$$\lim_{r \downarrow 0+} \int_r^R f(x) \, \frac{\ln(x) - \ln(r)}{\ln(R) - \ln(r)} \, dx \;=\; \int_0^R f(x) \, dx$$

5. Let $\mathcal{C} = C([0,1])$ be the space of continuous real-valued functions on the unit interval, with its standard topology; similarly, let $\mathcal{PL}$ denote the space of piecewise linear real-valued functions on $[0,1]$.

   (a) Given $\phi \in \mathcal{PL}$, a natural number n and $\varepsilon > 0$, show that there exists $\psi \in \mathcal{PL}$ whose right-hand derivative is everywhere greater than n in absolute value and such that for all $x \in [0,1]$ one has $|\phi(x) - \psi(x)\,| < \varepsilon$.

   (b) For each natural number n, let

   $$F_n = \{ f \in \mathcal{C} : \exists x_0 \, , x_0 \leq 1 - 1/n \text{ giving } |f(x) - f(x_0)\,| \leq n(x - x_0) \text{ when } x_0 \leq x < 1 \} \, .$$

   Show that F_n is a closed and nowhere dense subset of $\mathcal{C}$.

6. Let $\mathcal{H}$ be a Hilbert space and E, F be orthogonal projections to the closed subspaces Y, Z of $\mathcal{H}$, respectively. Let $E \vee F$ denote the orthogonal projection to $(Y^\perp \cap Z^\perp)^\perp$. Under the assumption that E and F commute, prove the following.

   (a) EF is the orthogonal projection to $Y \cap Z$.

   (b) $E \vee F = E + F - EF$.

   (c) $E \vee F$ is the orthogonal projection to $Y + Z$.