OSU Department of Mathematics
Qualifying Examination
Summer 2021

Linear Algebra

Instructions:

- Do **any four of the six problems**.
- Use separate sheets of paper for each problem. Clearly indicate the problem and page number (if several pages are used for a solution) on the top of the page.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete this examination.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination:
  1. Use a separate sheet of paper to clearly indicate your identification number and the four problems which you wish to be graded.
  2. Arrange your solutions according to the problem order with the problem selection selection page on top and any scratch-work on the bottom.
  3. Submit the exam:
     - **For the in-person exam:** place your solutions together with the selection sheet and scratch paper, in the order arranged as above, into the envelope in which you received the exam and submit it to the proctor.
     - **For the on-line exam:**
       * scan your exam in the order arranged as above, starting with the selection page and ending with the scratch-work, as a single pdf file (using e.g. CamScan phone app);
       * check that your scan is legible and contains all the necessary pages to be graded;
       * email the file directly to Nichole Sullivan (Nikki.Sullivan@oregonstate.edu);
       * wait online until it is confirmed that your submission was received.

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Common notation:

- $\mathcal{M}_{m,n}(\mathbb{F})$ is the set of all $m \times n$ matrices over a field $\mathbb{F}$. Here, $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.
- I_n is an $n \times n$ identity matrix.
- AB means either a product of two matrices or a composition of linear transformations, depending on the context.
- $\text{Range}(L)$ and $\text{Ker}(L)$ denote the range and the kernel (null-space) of a linear transformation L.
- $\text{tr}(A)$ denotes the trace of a square matrix A.
- $\det(A)$ denotes the determinant of a square matrix A.
- A^* (A is a matrix) is the adjoint matrix: $A^* = \bar{A}^t$.

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Problems:

1. (10pt) Let n be an odd positive integer. Let A be an $n \times n$ real orthogonal matrix such that $\det(A) > 0$. Prove that there is a nonzero vector v such that $Av = v$.

2. (10pt) Let A be an $n \times n$ real matrix. Suppose $\text{tr}(AX) = 0$ for any $n \times n$ real matrix X with $\text{tr}(X) = 0$. Prove that $A = \lambda I_{n\times n}$ for some $\lambda \in \mathbb{R}$.

3. Let $A, B \in \mathcal{M}_{m,n}(\mathbb{C})$ such that $\text{Range}(A^*) \bigcap \text{Range}(B^*) = \{0\}$ and $B^*A = 0$.

   (a) (2pt) Prove that $\text{Range}(A)$ is orthogonal to $\text{Range}(B)$.

   (b) (8pt) Prove that $\text{Rank}(A + B) = \text{Rank}(A) + \text{Rank}(B)$.

4. (10pt) Suppose $A \in \mathcal{M}_{n,n}(\mathbb{C})$ is such that A has an eigenspace of dimension bigger than one. Prove that for any $v \in \mathbb{C}^n$ the vectors v, Av, $\ldots$, $A^{n-1}v$ are linearly dependent.

5. (10pt) Suppose $L : V \to V$ − linear operator on an n-dimensional vector space V, L − nilpotent, i.e. $L^k = 0$ for some $k \in \mathbb{N}$. Prove that $L^n = 0$.

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6. (10pt) Consider a continuous function $f:\mathbb{R}\to(0,\infty)$ such that $\int\limits_{-\infty}^{\infty}|x|^j f(x)\,\mathrm{d}x<\infty$ for all $j\geq 0$. Denote by $\mathcal{P}_m$ the vector space of all real-valued polynomials of degree at most m, equipped with the inner product

$$\langle g,h\rangle=\int\limits_{-\infty}^{\infty} g(x)h(x)f(x)\,\mathrm{d}x\ .$$

(you may assume that the above defines a valid inner product). Let $m_j=\int\limits_{-\infty}^{\infty}x^j f(x)\,\mathrm{d}x$ $(j=0,1,2,\ldots)$ and consider polynomials $p_0(x)=m_0$ and

$$p_n(x)=\det\begin{pmatrix} m_0 & m_1 & m_2 & \ldots & m_n \\ m_1 & m_2 & m_3 & \ldots & m_{n+1} \\ m_2 & m_3 & m_4 & \ldots & m_{n+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{n-1} & m_n & m_{n+1} & \ldots & m_{2n-1} \\ 1 & x & x^2 & \ldots & x^n \end{pmatrix} \qquad n=1,2,\ldots$$

Prove that $p_0,\ldots,p_m$ are orthogonal in $\mathcal{P}_m$.