Department of Mathematics OSU
Qualifying Examination
Spring 2016

Complex Analysis and Linear Algebra

- Do any two of the three problems in Part CA, *use the corresponding marked blue book* and indicate on the selection sheet with your indentification code those problems which you want to have graded. Similarly, do any two of the three problems in Part LA in the *corresponding marked blue book* and mark those which you want to have graded on the selection sheet.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete Part II.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination, place examination blue book(s) and selection sheets back into the envelope in which the test materials came. You will hand in all materials. If you use extra examination books, be sure to place your code number on them and mark whether they are for *complex analysis* or *linear algebra*.

DO NOT WRITE YOUR NAME ANYWHERE – USE ONLY YOUR TEST ID CODE

Part CA: Complex Analysis

1. Determine all entire functions f such that $f(f(z)) = z$ for all $z \in \mathbb{C}$.

2. (a) Let $f(z)$ be analytic in a simply-connected region D containing a simple piecewise smooth closed contour C (oriented counterclockwise). Suppose $g(z)$ is also analytic in D and that $f(z)$ has no roots on C. Let $z_1, z_2, \ldots, z_n$ be all the zeros of $f(z)$ in the interior of C and let $m_1, m_2, \ldots, m_n$ be their respective algebraic multiplicities. Show that

$$\frac{1}{2\pi i} \oint_C g(z) \frac{f'(z)}{f(z)}\, dz = \sum_{j=1}^{n} m_j g(z_j) \ .$$

   (b) Let $f(z)$ be analytic on a simply-connected region containing $\bar{D}(0,2)$, the closed disk of radius two centered at the origin, and let C be the circle of radius two (centered at the origin) oriented counterclockwise. Given that f is non-zero on C and that

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)}\, dz = 2, \quad \frac{1}{2\pi i} \oint_C z \frac{f'(z)}{f(z)}\, dz = 2, \quad \frac{1}{2\pi i} \oint_C z^2 \frac{f'(z)}{f(z)}\, dz = -\frac{5}{2}$$

   find the zeros of f in $D(0,2)$.

3. Let f be analytic in the open disk $|z| < R$ and assume that there is a positive number M such that $|f'(z)| \le M < \infty$ for $|z| < R$. Show that in the expansion $f(z) = \sum_{n=0}^{\infty} f_n z^n$ one has, for $n \ge 1$,

$$|f_n| \le \frac{M}{nR^{n-1}}.$$

Exam continues on next page ...

Part LA: Linear Algebra

1. Suppose that A and B are two symmetric $n \times n$ real matrices and that A is positive definite (i.e. $\langle x, Ax\rangle > 0$ for all $0 \neq x \in \mathbb{R}^n$; equivalently, all eigenvalues of A are positive). Show that there is an invertible real matrix U such that U^tAU is the identity matrix and U^tBU is diagonal.

2. For arbitrary elements a, b and c in a field F, compute the minimal polynomial of the matrix
$$A = \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{pmatrix}.$$

3. Let T and S be linear transformations on a vector space V (not necessarily finite dimensional) whose kernels (null-spaces) are finite dimensional. Prove that the kernel of the composition $T \circ S$ is also finite dimensional and that
$$\dim \ker T \circ S \leq \dim \ker T + \dim \ker S.$$