Department of Mathematics OSU
Qualifying Examination
Spring 2018

Complex Analysis and Linear Algebra

- Do any two of the three problems in Part CA, and indicate on the selection sheet with your indentification code those problems which you want to have graded. Similarly, do any two of the three problems in Part LA in and mark those which you want to have graded on the selection sheet. Since we no longer require separate blue books for the two parts, please mark your answers with the CA or LA prefix.)
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete this exam.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination, place examination blue book(s) and selection sheets back into the envelope in which the test materials came. You will hand in all materials. If you use extra examination books, be sure to place your code number on them.

DO NOT WRITE YOUR NAME ANYWHERE – USE ONLY YOUR TEST ID CODE

Part CA: Complex Analysis

1. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Prove that f is one-to-one on the open unit disk if $\sum_{n=2}^{\infty} n|a_n| \leq 1$.

2. For all $z \in \mathbb{C} \setminus \mathbb{R}$ use the residue theorem to compute

$$g(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \left(\frac{1}{x-z} - \frac{x}{x^2+1}\right) dx.$$

3. Let $\mathcal{H}$ be the upper half-plane consisting of those $z = x + iy \in \mathbb{C}$ such that $y > 0$, and let f be an analytic map of $\mathcal{H}$ into itself.

   (a) Show that for any $\zeta \in \mathcal{H}$, $T_\zeta : z \to \dfrac{z-\zeta}{z-\overline{\zeta}}$ is an invertible analytic map of $\mathcal{H}$ onto the (open) unit disk D.

   (b) Let $z_0 \in \mathcal{H}$. Prove that for all $z \in \mathcal{H}$

$$\frac{|f(z) - f(z_0)|}{|f(z) - \overline{f(z_0)}|} \leq \frac{|z - z_0|}{|z - \overline{z_0}|}.$$

Exam continues on next page ...

Part LA: Linear Algebra

1. Consider three real quadratic polynomials,

$$f_i(x) = a_i x^2 + b_i x + c_i\,, \quad i = 1, 2, 3,$$

where a_1 is non-zero. Suppose that $b_1^2 - 4a_1c_1 \neq 0$ and that the graphs $y = f_1(x)$ and $y = f_2(x)$ share exactly one common point of intersection with the x-axis, which we label as P_0. Show that the graph $y = f_3(x)$ also passes through this point (i.e. P_0) if and only if the matrix

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

is **not** invertible.

2. Suppose that A is an $n \times n$ matrix whose characteristic polynomial splits and that γ a (Jordan) cycle of generalized eigenvectors for the eigenvalue λ, and let γ' be γ in reverse order (i.e. if $\gamma = \{w_1, w_2, \ldots, w_r\}$ then $\gamma' = \{w_r, w_{r-1}, \ldots, w_1\}$). Let W be the span of γ and T_W the restriction of the linear operator $T : v \mapsto Av$ to W.

   (a) Show that

$$[T_W]_{\gamma'} = (\,[T_W]_\gamma\,)^t.$$

   (b) Let J be the Jordan canonical form of A. Show that J and its transpose J^t are similar matrices.

   (c) Deduce that A and its transpose A^t are similar matrices.

3. Suppose that V is a finite dimensional vector space over a field k, and denote by $\mathcal{L}(V)$ the linear operators on V.

   (a) If $P : V \to V$ is a projection, show that the map

$$\begin{aligned} \mu_P : \mathcal{L}(V) &\to \mathcal{L}(V) \\ T &\mapsto PT \end{aligned}$$

   is a projection as a linear operator on the k-vector space $\mathcal{L}(V)$.

   (b) Now let $V = \mathbb{R}^n$, with its standard inner product. The linear operators also form an inner product space using the Frobenius inner product: $\langle A, B\rangle = \mathrm{tr}\,(B^tA)$, where tr denotes the trace, and B^t denotes the transpose (which here is the adjoint) of the linear operator B.

   Determine the set of projections P on $\mathbb{R}^n$ such that the corresponding maps μ_P are **orthogonal** projections for the Frobenius inner product.