Department of Mathematics OSU
Qualifying Examination
Fall 2010

PART I : Real Analysis

- Do any of the four problems in Part I. Indicate on the sheet with your identification number the four which you wish graded.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete Part I.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination, place test and examination book back into the envelope. If you use extra examination books, be sure to place your code number on them.

1. Let E be a Lebesgue measurable subset of $\mathbb{R}$ and let $f_n : E \to \mathbb{R}$ for all $n \geq 1$ be measurable functions.

Definition. The sequence $\{f_n\}$ converges on E to a function f *in measure* if and only if for every $\epsilon > 0$ there exists N such that $m(\{x \in E : |f_n(x) - f(x)| \geq \epsilon\}) < \epsilon$ for all $n \geq N$. Here, m denotes Lebesgue measure.

Now assume that $\{f_n\}$ converges pointwise to a function f on E, and that $m(E) < \infty$.

- Prove that $f_n \to f$ in measure on E.

Hint. Let $\epsilon > 0$. Work with the sets $E_n = \{x \in E : |f_n(x) - f(x)| < \epsilon\}$ and $G_N = \cap_{n=N}^{\infty} E_n$.

2. Let E be a compact subset of a metric space X.

a.) Show that E is a closed and bounded subset of X.

b.) Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space. Show that the closure of the unit ball in $\mathcal{H}$ is not a compact subset of $\mathcal{H}$.

Note. A proof is required. Merely citing a general theorem does not suffice.

3. Let $C([0,1])$ be the metric space of continuous functions on the interval $[0,1]$, equipped with the usual metric $\rho(f,g) = \max_{0 \leq x \leq 1} |f(x) - g(x)|$. For any positive integer n let F_n be the subset of $C([0,1])$ consisting of those functions f for which there is a point x_0 in $[0,1]$ (with x_0 depending on f) such that

$$|f(x) - f(x_0)| \leq n|x - x_0| \quad \text{for all } x \in [0,1].$$

a.) Show that F_n is nowhere dense in $C([0,1])$.

Hint. You may use without proof that for $f \in C([0,1])$ and $r > 0$ there exists a piecewise linear function $g \in C([0,1])$ such that both (1) $\rho(f,g) < r$ and, (2) the absolute value of the slope of each of the finitely many linear pieces of g is greater than or equal to $n+1$.

b.) Use part (a) to show that there exist functions in $C([0,1])$ that are nowhere differentiable in $(0,1)$.

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4. Let $g_\nu \in L^1(\mathbb{R}^n)$ be a sequence of functions such that

$$\sum_{\nu=1}^{\infty} \| g_\nu \|_{L^1} < \infty ,$$

where $\| f \|_{L^1} = \int_{\mathbb{R}^n} |f(x)| \, dx$. Show that

i) $\displaystyle\sum_{\nu=1}^{\infty} g_\nu$ converges a.e. to a function g, and that

ii) $\displaystyle\lim_{k\to\infty} \| g - \sum_{\nu=1}^{k} g_\nu \|_{L^1} = 0.$

Hint. Begin by showing that $\int_{\mathbb{R}^n} \sum_{\nu=1}^{\infty} |g_\nu(x)| \, dx < \infty$.

5. Assume $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$, $1 < p < \infty$, $1 < q < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$. Define the *convolution* of f and g by

$$h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y) \, dy,$$

where the integration is with respect to Lebesgue measure.

(a) Prove that $h(x)$ is well-defined for every $x \in \mathbb{R}$, and find a (constant) upper bound for $|h(x)|$.

(b) Prove that the function h is uniformly continuous on $\mathbb{R}$.

Hint. As part of your work, you will need to prove a result regarding translations in $L^p(\mathbb{R})$, and during that process you may use the fact (without proving it) that continuous functions with compact support are dense in $L^p(\mathbb{R})$ if $1 < p < \infty$.

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6. a) Let B be an $n \times n$ matrix, regard B as defining a linear operator on $\mathbb{R}^n$, and let $\|\cdot\|_2$ denote the usual Euclidean vector norm on $\mathbb{R}^n$, i.e., $\|x\|_2 = \left(\sum_{j=1}^{n} |x_j|^2\right)^{1/2}$ for all $x \in \mathbb{R}^n$. An operator norm of the matrix B can be defined by the relation

$$\|B\| \;=\; \sup_{x \neq 0} \frac{\|Bx\|_2}{\|x\|_2} \;=\; \sup_{\|x\|_2 = 1} \|Bx\|_2\,;$$

this definition implies $\|Bx\|_2 \leq \|B\| \|x\|_2$ for all x.

- Use properties related to metric spaces to prove that $\|B\| < \infty$ and that "sup" can be replaced by "max".

b.) Let A be a nonsingular $n \times n$ matrix. A linear system $Ax = b$ can be solved iteratively as follows.

(1) First write A in the form $A = M - N$, with M invertible and N non-zero; there are infinitely many ways to do this, so assume that one particular choice has been made.

(2) The system $Ax = b$ is then equivalent to $Mx = Nx + b$, or $x = Bx + c$, where $B = M^{-1}N$ and $c = M^{-1}b$.

(3) One can then try iteration of the form $x^{(k+1)} = Bx^{(k)} + c$, where $x^{(0)}$ is chosen arbitrarily.

However, this may or may not work, depending on the choice of B.

- State and prove a condition on B that is sufficient to guarantee that $x^{(k)} \to x$ as $k \to \infty$.