Department of Mathematics
Qualifying Examination
Fall 2003

Part I: Complex Analysis and Linear Algebra

- Do any two problems in Part CA and any two problems in Part LA.
- **After you have completed the exam**, fill out the separted cover sheet that specifies the problems you are submitting for evaluation by the examination committee.
- Your solutions should include all essential mathematical details; please write them up as clearly as possible.
- State explicitly any standard theorems that are needed to justify your reasoning.
- You have three hours to complete Part I of the exam.
- In problems with multiple parts, the individual parts may be weighted differently in grading.
- In general, an exam paper that gives partial solutions to most of the problems attempted, completing some parts of the problems attempted and skipping other parts, will not be graded as favorably as an exam that, for the most part, consists of complete solutions to the problems attempted.

Part CA

1. Find *all* functions with the following two properties:

   (a) $f(z)$ is analytic in $z \neq 0$.

   (b) $r^{-1}\left(\max_{|z|=r} |f(z)|\right) \to 1$ as $r \to 0$ and as $r \to \infty$.

2. Let $f(z)$ be an entire function, meaning that f is analytic in the entire complex plane, and C be a positively oriented simple closed curve in the complex plane. Let $Z_C(f)$ be the number of zeros (counted to multiplicity) of f in the interior of C.

   (a) Assume $f(z) \neq 0$ for z on C. Prove

   $$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = Z_C(f)$$

   (b) Assume $\{f_n(z)\}_{n=1}^{\infty}$ is a sequence of entire functions, each f_n has only real zeros, $f_n \to f$ uniformly on each compact subset of the complex plane, and f is not indentically zero. Prove that $f(z)$ is entire and can have only real zeros. (*Hint.* Part (a) may be helpful.)

3. Use appropriate methods of complex analysis to do the following:

   (a) Evaluate

   $$\int_0^{\infty} \frac{1}{x^4+1}\,dx$$

   (b) Let n be a positive integer and $0 < \theta < \pi$. The integral

   $$\frac{1}{2\pi i}\int_{|z|=\pi} \frac{z^n}{z^2 - 2z\cos\theta + 1}\,dz$$

   is real. Evaluate the integral and express your answer in terms of n, θ, and real trigonometric functions.

Part LA

1. Let a, b, c, d be real numbers that are not all zero. Let

$$A = \begin{bmatrix} aa & ab & ac & ad \\ ba & bb & bc & bd \\ ca & cb & cc & cd \\ da & db & dc & dd \end{bmatrix}$$

Find all the eigenvalues of A, the dimension of the eigenspace for each eigenvalue, and determine whether or not $\mathbb{R}^4$ is the direct sum of the eigenspaces of A. (Recall that a vector space V is the direct sum of subspaces $W_1, \ldots, W_k$ if each element v in V has a unique representation as $v = w_1 + \cdots + w_k$ with each w_j in W_j.)

2. The *rank* of an $n \times n$ matrix is defined to be the dimension of its column space. Let A and B be $n \times n$ complex matrices.

   (a) Prove that $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$.

   (b) Exhibit specific matrices A and B with $A \neq B$ such that

      i. $\text{rank}(AB) = \text{rank}(A) < \text{rank}(B)$

      ii. $\text{rank}(AB) = \text{rank}(B) < \text{rank}(A)$,

      iii. $\text{rank}(AB) < \min\{\text{rank}(A), \text{rank}(B)\}$

3. Let T be a linear operator on a real vector space and v be a nonzero vector. Let W be the T-cyclic subspace generated by v. Suppose that the dimension of W is 3, and that $T^4 v = 2T^3 v - T^2 v$.

   (a) Give all possible Jordan Canonical Forms for the restriction of T to W.

   (b) For each of the above, give a corresponding ordered Jordan basis, with elements expressed in terms of T and v.

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Part II: Real Analysis

- Do any four of the problems in Part II.
- **After you have completed the exam**, fill out the separted cover sheet that specifies the problems you are submitting for evaluation by the examination committee.
- Your solutions should include all essential mathematical details; please write them up as clearly as possible.
- State explicitly any standard theorems that are needed to justify your reasoning.
- You have three hours to complete Part II of the exam.
- In problems with multiple parts, the individual parts may be weighted differently in grading.
- In general, an exam paper that gives partial solutions to most of the problems attempted, completing some parts of the problems attempted and skipping other parts, will not be graded as favorably as an exam that, for the most part, consists of complete solutions to the problems attempted.

1. The zero set of a polynomial $p(x, y)$ in two variables is

$$\{(x, y) : p(x, y) = 0\}\,.$$

Show that $\mathbb{R}^2$ cannot be written as a countable union of zero sets of non-trivial polynomials.

2. Let $|\cdot|$ be the Euclidean norm on $\mathbb{R}^M$: If $x = (x_1, ..., x_M)$, then $|x| = \sqrt{\sum_{k=1}^{M} x_k^2}$. Suppose that for each $n = 1, 2, \ldots$ we have a set of $n+1$ points $p_{j,n} \in \mathbb{R}^M$, $j = 0, 1, 2 \ldots, n$, such that $|p_{0,n}| \leq B$ for a constant B independent of n and

$$|p_{j-1,n} - p_{j,n}| \leq \frac{1}{n}$$

holds for $j = 1, 2 \ldots, n$. Show that there is a continuous function $f : [0,1] \to \mathbb{R}^M$ such that, for each $t \in [0,1]$, $f(t)$ is in the closure of

$$P = \{p_{j,n} : n = 1, 2, \ldots ;\ j = 0, 1, 2 \ldots, n\}.$$

Hint. Consider the piecewise linear functions on $[0,1]$ that satisfy $f_n(j/n) = p_{j,n}$.

3. Do the following:

   (a) Suppose $f : A \to \mathbb{R}$ is continuous. Prove that the graph of f,

   $$\{(a, f(a)) : a \in A\},$$

   is a closed set.

   (b) Give an example of a function $g : \mathbb{R} \to \mathbb{R}$ such that the graph is closed, but the function is not continuous.

4. Let $\mathcal{B}_n$, $\mathcal{B}_m$, and $\mathcal{B}_{n+m}$ be the collection of all Borel sets in $\mathbb{R}^n$, R^m, and $\mathbb{R}^{n+m}$ respectively. Suppose $\mathcal{B}_n \times \mathcal{B}_m$ denotes the smallest $\sigma-$algebra of subsets of $\mathbb{R}^n \times R^m$ which contains all Borel rectangles, that is, all sets of the form $\{E_1 \times E_2 \mid E_1 \in \mathcal{B}_n,\ E_2 \in \mathcal{B}_m\}$.

   (a) Show that $\mathcal{B}_n \times \mathcal{B}_m = \mathcal{B}_{n+m}$.

   (b) Replace the Borel sets $\mathcal{B}_n$, $\mathcal{B}_m$, and $\mathcal{B}_{n+m}$ with the Lebesgue measurable sets $\mathcal{L}_n$, $\mathcal{L}_m$, and $\mathcal{L}_{n+m}$. Does the corresponding statement in (a) for Lebesgue measurable sets hold? If so explain why. If not explain what is true. (Justify your answers.)

5. Do the following:

   (a) State Fatou's lemma.

(b) Give an example that shows that the inequality in Fatou's lemma cannot be replaced by equality.

(c) Give two essentially different *additional* assumptions about the functions in Fatou's lemma that ensure equality will hold in the lemma.
Remark: In each case, your additional assumptions must give a result that applies to a reasonably general class of functions. An unacceptable answer would be "Let all the functions in Fatou's lemma be identically zero, then equality holds in Fatou's lemma."

(d) Prove one of the statements you made in (c).

(e) If "lim inf" is replaced by "lim sup" in Fatou's lemma, show that, in general, neither inequality $\leq$ nor $\geq$ is valid.

6. Do the following:

(a) If $f \in L^{p_0}(\mathbb{R}^n)$ for some $1 \leq p_0 < \infty$, prove that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

(b) Show that the same conclusion need not hold if f is measurable but the hypothesis $f \in L^{p_0}(\mathbb{R}^n)$ from some $1 \leq p_0 < \infty$ is omitted.