Department of Mathematics
Qualifying Examination
Fall 2005

Part I: Complex Analysis and Linear Algebra

- Do any two problems in Part CA and any two problems in Part LA.
- Your solutions should include all essential mathematical details; please write them up as clearly as possible.
- State explicitly including all hypotheses any standard theorems that are needed to justify your reasoning.
- You have three hours to complete Part I of the exam.
- In problems with multiple parts, the individual parts may be weighted differently in grading.

Part: Complex Analysis

1. Let $\mathcal{C}$ denote the entire complex plane. Suppose f is an non-constant entire function, meaning that f is analytic in the entire complex plane, from $\mathcal{C}$ to $\mathcal{C}$. Show that the range $f(\mathcal{C})$ is dense in $\mathcal{C}$.

2. Suppose $z = x + iy$ is a complex number where $x,\ y$ are real numbers. Denote $\text{Im}(z) = y$. Find a conformal map from the upper half open disk $\{z\ :\ |z| < 1,\ \text{Im}(z) > 0\ \}$ onto the open unit disk $\{z\ :\ |z| < 1\ \}$.

3. (a) Suppose a function f is analytic everywhere in the complex plane except for a finite number of singular points interior to a circle Γ, $|z| = r,\ r > 0$, which is transversed once in the counterclockwise direction. Suppose $\text{Res}\{f(z),\ z_0\}$ denotes the residue of $f(z)$ at z_0. Prove the equality

$$\oint_\Gamma f(z)\ dz = 2\pi i\ \text{Res}\left\{\frac{1}{z^2}f(\frac{1}{z}),\ 0\right\}.$$

   (b) Suppose $P(z)$ and $Q(z)$ are two complex polynomials with degree n and m respectively. Suppose Γ is a simple closed contour that encloses all zeros of $Q(z)$. If $m \geq n + 2$ show that the contour integral

$$\oint_\Gamma \frac{P(z)}{Q(z)}\ dz = 0.$$

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Part: Linear Algebra

1. Let $f : V \times V \to \mathbb{R}$ be a bilinear form on a finite-dimensional real vector space V. Thus f is linear in both variables:

$$\begin{aligned} f(ax+by, z) &= af(x,z) + bf(y,z) \\ f(x, cy+dz) &= cf(x,y) + df(x,z). \end{aligned}$$

Suppose that $v \in V$ is such that $f(v,v) \neq 0$. Let $\langle v \rangle$ be the subspace of V generated by v and let $v^\perp$ be the following subset of V:

$$v^\perp = \{x \in V : f(x,v) = 0\}.$$

(a) Prove that $v^\perp$ is a subspace of V.

(b) Prove that $V = \langle v \rangle \oplus v^\perp$.

(c) The function $R : V \to V$ given by:

$$R(x) = x - 2\frac{f(x,v)}{f(v,v)} \cdot v$$

is a linear transformation of V. What is the Jordan canonical form of (a matrix representation for) R?

(d) What is the determinant of (a matrix representation for) R?

2. An $n \times n$ matrix A is nilpotent if $A^m = 0_n$ for some $m \geq 1$, where 0_n is the $n \times n$ zero matrix.

(a) Find $\det(I_n + A)$. Here I_n is the identity $n \times n$-matrix.

(b) Show that if A is a nilpotent $n \times n$ matrix, then $A^n = 0$.

3. Let A and B nonsingular square complex matrices and suppose that $AB = BA^2$.

(a) Prove that if λ is an eigenvalue of A, then $\lambda^m = 1$ for some m.

(b) Prove that for some $n \geq 1$, A and B^n have a common eigenvector.

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Part II: Real Analysis

- Do any four of the problems in Part II.
- Your solutions should include all essential mathematical details; please write them up as clearly as possible.
- State explicitly including all hypotheses any standard theorems that are needed to justify your reasoning.
- You have three hours to complete Part II of the exam.
- In problems with multiple parts, the individual parts may be weighted differently in grading.

1. Assume that f and g are in $L^1(\mathbf{R})$ and $\int_E f\,d\mu = \int_E g\,d\mu$ for every Lebesgue-measurable set $E \subset \mathbf{R}$. (Here, μ denotes Lebesgue measure.) Prove $f = g$ almost everywhere in $\mathbf{R}$.

2. Let $F(t) = \int_{-\infty}^{\infty} f(x,t)\,dx$ for all t in some interval I. It would then be useful to say
$$F'(t) = \int_{-\infty}^{\infty} \frac{\partial f}{\partial t}(x,t)\,dx$$
for all $t \in I$. State hypotheses for which this statement is true, and prove this statement. Use hypotheses and a proof that have a "real analysis", or "Lebesgue", flavor. Partial credit will be given for using "advanced calculus" concepts such as uniform continuity and uniform convergence. (Give hypotheses which apply to a broad class of functions; for example, do not just assume something like $\partial f/\partial t = 0$.)

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3. (a) Let f be a continuous real-valued function on $[0,\ 1]$. Show that there exists a sequence of polynomials $p_1,\ p_2,\ \cdots$ such that

$$\lim_{n\to\infty} p_n(x^2) = f(x)$$

uniformly on $[0,\ 1]$.

(b) Show, by way of example, that no such sequence of polynomials may exist if the interval $[0,\ 1]$ is replaced by $[-1,\ 1]$.

4. (a) State a sequence $f_1,\ f_2,\ f_3,\ \ldots$ of Riemann-integrable functions on $[0,1]$ which is a Cauchy sequence in $L^1[0,1]$ but does *not* converge in L^1 to any Riemann-integrable function. Prove that your example has the required properties.

(b) Prove or give a counterexample of the following statement. Suppose f_1, $f_2,\ f_3,\ \ldots$ is a sequence of of Lebesgue-integrable functions on $[0,1]$ which converges pointwise to a function f on $[0,1]$, almost everywhere. Then

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\ dx.$$

If you give a counterexample, prove that your counterexample has the required properties.

5. Let $\langle X, \rho\rangle$ be a compact metric space and $\beta > 0$. A real-valued function f on X is said to be *uniformly β-continuous* if f has no jumps greater than β. To be precise, for each $\epsilon > 0$ there is a $\delta > 0$ so that $\rho(x, y) < \delta$ implies $|f(x) - f(y)| < \beta + \epsilon$. Suppose that $f_n \to f$ uniformly on X and that f_n is β_n-continuous with $\beta_n = 1/n$, for each $n \geq 1$. Must f be continuous? Prove your assertion.

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6. The differential equation $y'(x) = f(x, y(x))$ for $0 < x < 1$, together with the initial condition $y(0) = y_0$, can be written in the integral form

$$y(x) = y_0 + \int_0^x f(t, y(t))\, dt$$

for $0 \leq x \leq 1$. Assume that there exists a constant L, with $0 < L < 1$, such that $|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$ for all $t \in [0, 1]$ and all real y_1 and y_2. Prove that, for given y_0, there exists a unique function y in $C[0, 1]$ which satisfies the above integral form. (You may use, without proof, the fact that $C[0, 1]$ is complete with respect to the norm $\|\cdot\|_\infty$ defined by $\|f\|_\infty = \max_{0\leq x\leq 1} |f(x)|$.)