**Department of Mathematics**
**Qualifying Examination**
**Fall 2006**

**Part I: Real Analysis**

- Do any four of the problems in Part I.
- Your solutions should include all essential mathematical details; please write them up as clearly as possible.
- State explicitly including all hypotheses any standard theorems that are needed to justify your reasoning.
- You have three hours to complete Part I of the exam.
- In problems with multiple parts, the individual parts may be weighted differently in grading.

1. The Fourier transform of a function $f \in L^1(\mathbf{R})$ is the function $\hat{f}$ defined by $\hat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-iyx}dx$.

   (a) Prove that the function $\hat{f}$ is continuous on $\mathbf{R}$, for all $f \in L^1(\mathbf{R})$.

   (b) Prove that if $f \in L^1(\mathbf{R})$ and $f$ has compact support, then $\hat{f} \in C^\infty(\mathbf{R})$. As part of your solution, you will need to justify differentiation under the integral sign. (Here, "$f$ has compact support" means that there exists a compact set $K \subset \mathbf{R}$ such that $f(x) = 0$ for all $x \notin K$, and "$\hat{f} \in C^\infty(\mathbf{R})$" means that $\hat{f}$ has continuous derivatives of all orders.)

2. A point $x$ in a metric space is called isolated if the set $\{x\}$ is open.

   (a) Prove that a point $x$ in a metric space $X$ is isolated if and only if there exists $\epsilon > 0$ such that $\rho(x, y) \geq \epsilon$ for all $y \in X$ with $y \neq x$. Here, $\rho$ is the metric on $X$.

   (b) Prove that a complete metric space without isolated points has an uncountable number of points.

3. Let $f \in L^1(\mathbf{R})$, and for each positive integer $n$ define $f_n(x) = f\left(x + \frac{1}{n}\right)$ for all real $x$. Prove that $f_n \to f$ in $L^1(\mathbf{R})$ as $n \to \infty$.

   *Hint.* You may use the fact, without proving it, that the set of continuous functions with compact support is dense in $L^1(\mathbf{R})$.

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4. Assume that a real-valued function $g$ is continuous and differentiable on an open interval containing $[0,1]$, and assume that there exist constants $c_1$ and $c_2$ so that $0 < c_1 \le g'(x) \le c_2$ for all $x \in [0,1]$. Prove $\mu(g(E)) = \int_E g'(x)\,dx$ for every Lebesgue-measurable set $E \subset [0,1]$. Here, $\mu$ denotes Lebesgue measure, and $g(E)$ denotes the set of all $g(x)$ for $x \in E$.

   *Hint.* Start with open intervals.

5. Prove that the space $L^\infty[0,1]$ is not separable, when regarded as a metric space with the metric induced by the norm $\|\cdot\|_\infty$ (essential supremum, or, essentially bounded.)

   *Hint.* Use functions $f_\alpha$ defined by $f_\alpha(x) = 1$ if $x \le \alpha$ and $f_\alpha(x) = 0$ if $x > \alpha$.

6. For $f \in L^1(\mathbf{R}^2)$ one defines the Radon transform of $f$ as follows.

$$Rf(\theta, s) = \int_{-\infty}^{\infty} f(s\theta + t\theta^\perp)\,dt$$

   where $\theta$ and $\theta^\perp$ are two unit vectors orthogonal to each other and $s \in \mathbf{R}$. In other words, $Rf(\theta, s)$ is the integral of $f$ over the line with direction $\theta^\perp$ whose closest point to the origin is equal to $s\theta$.

   When solving this problem you may assume without proof that the Lebesgue integral $\int_{R^2} f(x)\,dx$ is invariant under rotations of the coordinate system.

   (a) Let $\theta$ be given. Show that $Rf(\theta, s)$ exists for almost every $s \in \mathbf{R}$.

   (b) The (n-dimensional) Fourier transform of a function $f \in L^1(\mathbf{R}^n)$ is the function $\hat{f}$ defined by $\hat{f}(y) = \int_{R^n} f(x) e^{-i<x,y>} dx$, where $< x, y >= \sum_{i=1}^n x_i y_i$ denotes the inner product of $x$ and $y$.

   For fixed $\theta$ let $g(s) = Rf(\theta, s)$. Prove the following relation between the (one-dimensional) Fourier transform of $g$ and the (two-dimensional) Fourier transform of $f$ :

$$\hat{g}(\sigma) = \hat{f}(\sigma\theta), \quad \text{for all } \sigma \in \mathbf{R}.$$

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**Part II: Complex Analysis and Linear Algebra**

- Do any two problems in Part CA and any two problems in Part LA.
- Your solutions should include all essential mathematical details; please write them up as clearly as possible.
- State explicitly including all hypotheses any standard theorems that are needed to justify your reasoning.
- You have three hours to complete Part II of the exam.
- In problems with multiple parts, the individual parts may be weighted differently in grading.

## Part: Complex Analysis

1. (a) Find the Laurent series of $\frac{1}{z^2(1-z)}$ when $0 < |z| < 1$.
   (b) Find the Laurent series of $\frac{1}{z^2(1-z)}$ when $1 < |z| < \infty$.
   (c) Find the Laurent series of $\frac{1}{z^2(1-z)}$ when $0 < |z-1| < 1$.

2. Suppose that a function $f$ is analytic inside and on a positively oriented simple closed contour $C$ and that it has no zero on $C$. Show that if $f$ has $n$ zeros, $z_1$, $z_2$, $\cdots$ , $z_n$ inside $C$, where each $z_k$ is of multiplicity $m_k$, then

$$\oint_C \frac{z^3 f'(z)}{f(z)}\, dz = 2\pi i \sum_{k=1}^{n} m_k z_i^3.$$

3. Use the residue theorem to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{1}{1+x^{2n}}\, dx$$

where $n$ is a positive integer.

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## Part: Linear Algebra

1. Consider the set $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \,|\, a, b \in \mathbb{Q}\}$, with addition and multiplication induced by $\mathbb{Q}(\sqrt{2}) \subset \mathbb{R}$.

   (a) Show that $\mathbb{Q}(\sqrt{2})$ is a vector space over $\mathbb{Q}$, with (ordered) basis $\mathcal{B} = (1, \sqrt{2})$.

   (b) Let $\alpha = x + y\sqrt{2}$ be an element of $\mathbb{Q}(\sqrt{2})$. Define the function

   $$\begin{aligned} T_\alpha : \mathbb{Q}(\sqrt{2}) &\to \mathbb{Q}(\sqrt{2}) \\ v &\mapsto \alpha v \,, \end{aligned}$$

   where $\alpha v$ denotes the multiplication of $v$ by $\alpha$. Show that $T_\alpha$ is a linear transformation.

   (c) Determine the set of $\alpha \in \mathbb{Q}(\sqrt{2})$ such that the characteristic polynomial of $T_\alpha$ does *not* equal the minimal polynomial of $\alpha$.

2. Let $A$ be a real $n \times n$ matrix. Let $M$ denote the maximum absolute value of the eigenvalues of $A$.

   (a) Give a statement of the Spectral Theorem.

   (b) Prove that if $A$ is symmetric, then for all $x \in \mathbb{R}^n$, the Euclidean norm of $Ax$ is at most $M$ times the Euclidean norm of $x$.

   (c) Show that this is false in general when the restriction *symmetric* is removed.

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3. Suppose that $T$ is a linear transformation on a complex finite dimensional vector space $V$ with distinct eigenvalues.

   (a) Suppose the dimension of the vector space $V$ is $n$. Show that there is a cyclic vector for $T$, that is a vector $w$ so that $\{w,\ Tw,\ T^2w,\ \cdots,\ T^{n-1}w\}$ forms a basis for $V$.

   (b) Give a complete description of the matrix that represents $T$ with respect to this basis for $V$.

**Remark:** You may use the following theorem **without** giving a proof.
Theorem: Suppose $a_1,\ a_2,\ \cdots,\ a_n$ are $n$ distinct complex numbers. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & \cdots & a_n^2 \\ a_1^3 & a_2^3 & a_3^3 & \cdots & \cdots & a_n^3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \cdots & \cdots & a_n^{n-1} \end{pmatrix}$$

Then the determinant of the matrix, $A$, is not zero.