Department of Mathematics
Qualifying Examination
Fall 2007

Part I: Real Analysis

- Do any four of the problems in Part I.
- Your solutions should include all essential mathematical details; please write them up as clearly as possible.
- State explicitly including all hypotheses any standard theorems that are needed to justify your reasoning.
- You have three hours to complete Part I of the exam.
- In problems with multiple parts, the individual parts may be weighted differently in grading.

1. Suppose $f \in L^{p_0}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ holds, for some $1 \leq p_0 < \infty$.

   (a) Prove that $f \in L^p(\mathbb{R}^n)$ for p with $p_0 < p < \infty$.

   (b) Prove that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

   (c) Give an example of a function in $L^{p_0}(\mathbb{R}^n)$, for some $1 \leq p_0 < \infty$, but which is not in $L^p(\mathbb{R}^n)$, for some p with $p_0 < p < \infty$.

2. Let m^* denote Lebesgue outer measure on $\mathbb{R}$. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(x) - f(y)| \leq C\,|x - y|$$

   for all $x, y \in \mathbb{R}$ (here $0 < C < \infty$).

   (a) Prove that $m^*[f(A)] \leq C\, m^*(A)$ holds for every $A \subset \mathbb{R}$.

   (b) Prove that if $A \subset \mathbb{R}$ is a Lebesgue measurable set, then so is $f(A)$.

3. (a) Show that the metric space of continuous functions on the interval $[0, 1]$ equipped with the L^2-metric is incomplete.

   (b) By the diameter of a subset A of a metric space X is meant the number

$$d(A) = \sup_{x,y\in A} \rho(x, y).$$

   where ρ denotes the metric. Suppose X is complete, and let $\{A_n\}$ be a sequence of closed nonempty subsets of X *nested* in the sense that

$$A_1 \supset A_2 \supset \ldots \supset A_n \supset \ldots.$$

   Suppose further that

$$\lim_{n\to\infty} d(A_n) = 0.$$

   Prove that the intersection $\bigcap_{n=1}^\infty A_n$ consists of a single point.

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4. Let M be a bounded subset of $C([a, b])$. Consider the set $S \subset C([a, b])$ of all functions F such that

$$F(x) = \int_a^x f(t)\, dt$$

for some f in M. Show that the closure of S is a compact subset of $C([a, b])$. When solving this problem, state precisely the hypothesis and conclusion of any major theorem that you are using.

5. (a) Let $f : [-1, 1] \to \mathbb{R}$ be continuous. Suppose $\int_{-1}^{1} f(x)\, x^n\, dx = 0$ holds for $n = 0, 1, 2, \ldots$.. Prove that $f(x) = 0$ holds for all $x \in [-1, 1]$.

   (b) Let φ_n, $n = 1, 2, \ldots$ be a sequence of functions in $L^2([0, 2\pi])$ such that $\int_0^{2\pi} \varphi_n(t)\, \varphi_m(t)\, dt$ is equal to 1 for $n = m$ and vanishes for $n \neq m$. If $A \subset [0, 2\pi]$ and A is measurable, prove that

$$\lim_{n\to\infty} \int_A \varphi_n(x)\, dx \;=\; 0.$$

6. Let $\{ q_i \}_{i=0}^{\infty}$ be an enumeration of the rationals in the unit interval $[0, 1]$. Suppose that $q_0 = 0$ and $q_1 = 1$. Define a function f on the rationals in the unit interval by setting $f(q_0) = 0$, setting $f(q_1) = 1$, and, for $n \geq 2$, recursively setting

$$f(q_n) = \frac{f(q_n^-) + f(q_n^+)}{2}$$

where

$$\begin{aligned} q_n^- &= \max\{ q_i : i = 0, 1, \ldots, n-1 \;\text{ and }\; q_i < q_n \}, \\ q_n^+ &= \min\{ q_i : i = 0, 1, \ldots, n-1 \;\text{ and }\; q_n < q_i \}. \end{aligned}$$

So for instance, $q_2^- = q_0$, $q_2^+ = q_1$, and $f(q_2) = [\, f(q_0) + f(q_1)\,]/2 = 1/2$.

   (a) Prove that f is monotone on the rationals in $[0, 1]$.

   (b) Prove that f is continuous on the rationals in $[0, 1]$.

   (c) Can f be extended continuously to all real numbers in $[0, 1]$, and why or why not?

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Part II: Complex Analysis and Linear Algebra

- Do any two problems in Part CA and any two problems in Part LA.
- Your solutions should include all essential mathematical details; please write them up as clearly as possible.
- State explicitly including all hypotheses any standard theorems that are needed to justify your reasoning.
- You have three hours to complete Part II of the exam.
- In problems with multiple parts, the individual parts may be weighted differently in grading.

Part: Complex Analysis

1. Suppose f is an analytic function on the unit disc, $D \equiv \{|z| \leq 1\}$. Suppose $f(0) = 0$ and $|f(z)| \leq 1$, for all $z \in D$. Show that $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$, for all $z \in D$.

2. Let $P_n(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!}$. Show that for every given positive real number $r > 0$, there exists a positive integer M such that for every $n \geq M$ all zeros of the polynomial $P_n(z)$ lie outside the circle $|z| = r$.

3. Suppose f is a complex function defined on the open unit disc, $|z| < 1$.
   (a) Show or give a counterexample: If f^2 is analytic on D, then f is analytic on D.
   (b) Show that if f^2 and f^3 are analytic on D, then f is analytic on D.

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Part: Linear Algebra

1. Let F be a field. For m and n positive integers, let $M_{m,n}$ be the vector space of $m \times n$ matrices over F. Fix m and n, and fix matrices A and B in $M_{m,n}$. Define the linear transformation T from $M_{n,m}$ to $M_{m,n}$ by $T(X) = A\,X\,B$. Prove that if $m \neq n$, then T is not invertible.

2. Let S be the subspace of $M_{n,n}$ (the vector space of all real $n \times n$ matrices) generated by all matrices of the form $A\,B - B\,A$ with A and B in $M_{n,n}$. Prove that $\dim(S) = n^2 - 1$.

3. Let

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

   (a) Find the minimal and characteristic polynomials of M.

   (b) Is M similar to a diagonal matrix D over $\mathbb{R}$? If so, find such a D.

   (c) Repeat part (b) with $\mathbb{R}$ replaced by $\mathbb{C}$ and also by the field $\mathbb{Z}/5\mathbb{Z}$.