**OSU Department of Mathematics**
**Qualifying Examination**
**Spring 2021**

**Linear Algebra**

**Instructions:**

- Do **any four of the six problems**.
- Use separate sheets of paper for each problem. Clearly indicate the problem and page number (if several pages are used for a solution) on the top of the page.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete this examination.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination:
  1. Use a separate sheet of paper to clearly indicate your identification number and the four problems which you wish to be graded.
  2. Arrange your solutions according to the problem order with the problem selection selection page on top and any scratch-work on the bottom.
  3. Submit the exam:
     - **For the in-person exam:** place your solutions together with the selection sheet and scratch paper, in the order arranged as above, into the envelope in which you received the exam and submit it to the proctor.
     - **For the on-line exam:**
       * scan your exam in the order arranged as above, starting with the selection page and ending with the scratch-work, as a single pdf file (using e.g. CamScan phone app);
       * check that your scan is legible and contains all the necessary pages to be graded;
       * email the file directly to Nichole Sullivan (Nikki.Sullivan@oregonstate.edu);
       * wait online until it is confirmed that your submission was received.

**Exam continues on next page ...**

**Common notation:**

- $\mathcal{M}_{m,n}(\mathbb{F})$ is the set of all $m \times n$ matrices over a field $\mathbb{F}$. Here, $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.
- $I_n$ is an $n \times n$ identity matrix.
- $AB$ means either a product of two matrices or a composition of linear transformations, depending on the context.
- $\mathrm{Range}(L)$ and $\mathrm{Ker}(L)$ denote the range and the kernel (null-space) of a linear transformation $L$.
- $A^*$ ($A$ is a matrix) is the adjoint matrix: $A^* = \bar{A}^t$.
- $L^*$ ($L$ is a linear transformation $L$ between inner product spaces $U$ and $V$) is the adjoint transformation: $\forall\, u \in U,\, v \in V,\, \langle Lu, v\rangle_V = \langle u, L^* v\rangle_U$.
- $M^\perp$ is the orthogonal complement of a set $M$ in an inner product space $V$: $M^\perp = \{v \in V :\ \forall\, m \in M\ \langle m, v\rangle_V = 0\}$.

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**Problems:**

1. (10pt) A matrix $U \in \mathcal{M}_{n,n}(\mathbb{C})$ is called unitary if $UU^* = I_n$. Recall that unitary matrices are diagonalizable. Prove that if unitary matrices $U_1$ and $U_2$ in $\mathcal{M}_{n,n}(\mathbb{C})$ commute (i.e., $U_1U_2 = U_2U_1$), then they are *simultaneously diagonalizable*, i.e. there exists an orthonormal basis of $\mathbb{C}^n$ in which booth matrices, $U_1$ and $U_2$, are diagonal.

2. Let $V, W$ be finite dimensional vector spaces over a field $\mathbb{F}$, and $p : V \to W$ be an $\mathbb{F}-$linear map. Let $V'$ and $W'$ be the *dual* spaces of $V$ and $W$, i.e. the the vector spaces of all linear functionals: $V' = \{\phi : V \to \mathbb{F} : \ \phi \text{ is linear}\}$ together with the operations of function-addition and scalar multiplication, and similarly for $W'$. Moreover, consider a *dual map* $p' : W' \to V'$, defined by $p'(\phi)(v) = \phi(p(v))$ for any $\phi \in W'$ and $v \in V$.

   (a) (3pt) Briefly explain why $p'$ is a well-defined linear transformation.

   (b) (7pt) Prove that $p$ is surjective if and only if $p'$ is injective.

3. Suppose $A \in \mathcal{M}_{m,n}(\mathbb{C})$ and $B \in \mathcal{M}_{n,m}(\mathbb{C})$.

   (a) (2pt) First, assume $n = m$. Prove that $AB$ and $BA$ are similar matrices provided either $A$ or $B$ is invertible. Show that if both $A$ and $B$ are not invertible, then $AB$ is not necessarily similar to $BA$.

   For the rest of the problem, assume the dimensions $\underline{m, n \in \mathbb{N} \text{ are arbitrary}}$.

   (b) (2pt) Show that if $\lambda \neq 0$ is an eigenvalue of $AB$ then it is also an eigenvalue of $BA$.

   (c) (3pt) Prove that the eigenspaces of $AB$ and $BA$ corresponding to an eigenvalue $\lambda \neq 0$ have the same dimensions.

   (d) (3pt) Prove that the characteristic polynomials for matrices $AB$ and $BA$ satisfy

   $$\lambda^n p_{AB}(\lambda) = \lambda^m p_{BA}(\lambda) \qquad \forall\, \lambda \in \mathbb{C}.$$

   *Hint:* one way is to continue with ideas from (c) using Jordan forms, alternatively, you might try to use the following block matrices: $P = \begin{pmatrix} \lambda I_m & A \\ B & I_n \end{pmatrix}$ and $Q = \begin{pmatrix} I_m & 0 \\ -B & \lambda I_n \end{pmatrix}$.

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4. Let $A \in \mathcal{M}_{n,n}(\mathbb{C})$ is a matrix with entrees $a_{ij}$, $i, j \in \{1, \dots, n\}$.

   (a) (6pt) Prove that if $\lambda \in \mathbb{C}$ is an eigenvalue of $A$, then there exists $k \in \{1, \dots, n\}$ such that
   $$|\lambda - a_{kk}| \le \sum_{j \ne k} |a_{kj}|.$$

   (b) (4pt) Use (a) to prove that for any complex polynomial $p(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1 z + a_0$, all the roots are located in the disk $D = \{z \in \mathbb{C} : \ |z| \le R\}$, where $R = \max\{|a_0|, 1 + |a_1|, \dots, 1 + |a_{n-1}|\}$.

   *Hint:* The following block-matrix might prove useful:
   $$A = \begin{pmatrix} 0 & -a_0 \\ & -a_1 \\ I_{n-1} & \vdots \\ & -a_{n-1} \end{pmatrix}$$

5. Suppose $U, V, W$ – finite-dimensional inner product spaces over $\mathbb{C}$, and $L : U \to V$, $T : V \to W$ are linear transformations. Let $L^*$ and $T^*$ be adjoint transformations corresponding to $L$ and $T$.

   (a) (3pt) Prove that $\mathrm{Range}(L) = \mathrm{Ker}(L^*)^{\perp}$.

   (b) (7pt) Assume that the sequence $U \xrightarrow{L} V \xrightarrow{T} W$ is exact, i.e. $\mathrm{Range}(L) = \mathrm{Ker}(T)$. Consider the linear transformation
   $$S = LL^* + T^*T.$$
   Prove that $S$ is invertible.

6. (10pt) Consider the following space of functions
   $$V = \big\{ (a_2x^2 + a_1x + a_0)e^{2x} \ : \ a_0, a_1, a_2 \in \mathbb{R} \big\}.$$
   Then, $\mathcal{A} : V \to V$ defined by $\mathcal{A}f = f''$, where $f''$ denotes second derivative, is a linear operator. Find a Jordan basis of $V$ for $\mathcal{A}$ as well as the corresponding Jordan canonical form, that is, a basis and the matrix $J_{\mathcal{A}}$ of $\mathcal{A}$ with respect to this basis such that $J_{\mathcal{A}}$ is a Jordan canonical form.