Department of Mathematics OSU
Qualifying Examination
Fall 2020

Linear Algebra

- Do any of the four of the six problems. Indicate on the sheet with your identification number the four which you wish to be graded.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete this examination.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination, place examination blue book(s) and selection sheet into the envelope in which you received the exam. You will hand in all materials. If you use extra examination books, be sure to place your code number on them.

1. (a) (4pt) The rank of a complex square matrix is defined to be the dimension of its column space. Give a direct proof that $rank(A+B) \leq rank(A)+rank(B)$.

   (b) (6pt) For a complex $n \times 1$ column vector u with the Euclidean norm $\|u\| = 1$, an elementary orthogonal projector is the $n \times n$ matrix $Q = I - uu^*$, where $u^* = \overline{u}^t$. Prove that $rank(Q) = n - 1$.

2. Let A be an $n \times n$ real matrix.

   (a) (2pt) Briefly explain why there exist an orthogonal matrix B and an (upper) triangular matrix C such that $A = BC$.

   (b) (6pt) Prove the Hadamard inequality:

   $$|\det(A)| \leq \prod_{j=1}^{n} \|a_j\|,$$

   where a_j denotes the j-th column of A.

   (c) (2pt) Show that the equality in (b) holds if and only if A is an orthogonal matrix or $a_j = 0$ for some j.

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3. A real symmetric $n \times n$ matrix A is called positive semi-definite if $x^t A x \geq 0$ for all $x \in \mathbb{R}^n$. Prove that A is positive semi-definite if and only if $tr(AB) \geq 0$ for every real symmetric positive semi-definite $n \times n$ matrix B.

4. We denote by $\mathcal{M}_{n\times n}(\mathbb{R})$ the set of all $n \times n$ real matrices.

   (a) (2pt) Prove that identity $(a+b)^{-1} = a^{-1} + b^{-1}$ can never hold for numbers a and b, yet it can hold for some $a, b \in \mathbb{C}$;

   (b) (6pt) Prove that there exist $A, B \in \mathcal{M}_{n\times n}(\mathbb{R})$ be such that $(A+B)^{-1} = A^{-1} + B^{-1}$ if and only if there exists $J \in \mathcal{M}_{n\times n}(\mathbb{R})$ such that $J^2 = -I_n$.

   (c) (2pt) As a consequence of (b), show that such matrices A and B exist if and only if n is even.

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5. Let W be a subspace of an inner product space $(V, \langle \cdot, \cdot \rangle)$ and $P : V \to V$ be a linear transformation. Prove that P is an orthogonal projection onto W if and only if P is both idempotent and self-adjoint.

   Recall: P is

   - *idempotent* if $P^2 = P$;
   - *self-adjoint* if $\forall\ x, y \in V\ \ \langle x, Py \rangle = \langle Px, y \rangle$;
   - *orthogonal projection onto* W if for any $x \in W$, $P(x) = x$ and for any $y \perp W$, $P(y) = 0$.

6. Let A be an $n \times n$ matrix over the real numbers with (complex) eigenvalues λ_1, λ_2, $\dots$, λ_n counted with multiplicity. Show that for any polynomial $q(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ the eigenvalues of $q(A)$ (counted with multiplicity) are $q(\lambda_1)$, $q(\lambda_2)$, $\dots$, $q(\lambda_n)$.