Department of Mathematics OSU
Qualifying Examination
Fall 2015

PART I : Real Analysis

- Do any four of the six problems in Part I. Indicate on the sheet with your identification number the four which you wish graded.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete Part I.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination, place examination blue book and selection sheet into the unmarked, smaller envelope. You will hand in all materials. If you use extra examination books, be sure to place your code number on them.

1. Let $\|\cdot\|$ be a norm on $\mathbf{R}^n$. Do not assume any properties of $\|\cdot\|$, other than those that follow from the general definition of norm on a vector space.

   (a) Let $f(x) = \|x\|$ for all $x \in \mathbf{R}^n$. Show that f is continuous on $\mathbf{R}^n$ with respect to the metric ρ defined by $\|\cdot\|$, i.e., $\rho(x,y) = \|x-y\|$ for all x and y in $\mathbf{R}^n$. (Use the triangle inequality.)

   (b) Now define a different norm $\|\cdot\|_1$ by $\|x\|_1 = \sum_{i=1}^n |x_i|$ for all $x \in \mathbf{R}^n$. Prove that the function f defined in part (a) is continuous with respect to the metric defined by $\|\cdot\|_1$. (You do not need to prove that $\|\cdot\|_1$ satisfies all of the axioms of a norm.)

   (c) Show that the norms $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent. That is, show that there exist positive constants M_1 and M_2 such that $M_1\|x\|_1 \leq \|x\| \leq M_2\|x\|_1$ for all $x \in \mathbf{R}^n$. (*Hint.* Consider what happens when f is restricted to the set $S = \{x \in \mathbf{R}^n : \|x\|_1 = 1\}$.)

   (d) Give an example of a linear space (vector space) X and two norms on X that are *not* equivalent, in the sense defined in part (c).

2. Define the convolution of two functions f and g by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy = \int_{-\infty}^{\infty} f(y)g(x-y)dy,$$

   assuming that the integrals exist. Let ϕ be a continuous function on $\mathbf{R}$ that satisfies $\phi(x) > 0$ for $-1 < x < 1$, $\phi(x) = 0$ otherwise, and $\int_{-\infty}^{\infty} \phi(x)dx = 1$. For each integer $n \geq 1$, let $\phi_n(x) = n\phi(nx)$ for all real x. Then $\int_{-\infty}^{\infty} \phi_n(x)dx = 1$ for all n, ϕ_n is nonzero on the interval $(-1/n, 1/n)$, and as n increases the graph of ϕ_n becomes narrow and tall. The convolution $(f * \phi_n)(x)$ is therefore a weighted average of values of $f(y)$ for y near x.

   Prove that if $f \in L^1(\mathbf{R})$, then $f * \phi_n \to f$ in $L^1(\mathbf{R})$ as $n \to \infty$. (That is, $\|f * \phi_n - f\|_1 \to 0$ as $n \to \infty$.)

   Suggestion. First consider the case where f is continuous and has compact support, and then extend to $L^1(\mathbf{R})$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \int_0^{\pi} \frac{\sin(xt)}{t} dt.$$

   Prove that f is a bounded function.

4. Consider the following minimization problem in $L^p[0,1]$, where $1 \leq p \leq \infty$:

$$\min_{u \in U} J(u), \quad J(u) := \int_0^1 f(u(s))ds,$$

where $f : \mathbb{R} \to [0, +\infty)$ is continuous, and U is a nonempty, compact subset of $L^p[0,1]$.

The goal is to prove the existence of a minimizer u^* in U for the functional J.

[**Note**: Part (a) and (b) can be solved separately, but part (b) requires the use of the result in part (a).]

(a) Prove that J is lower semicontinuous: if $u_n \to u$ in $L^p[0,1]$, then

$$J(u) \leq \liminf_n J(u_n). \tag{1}$$

(b) Show that a minimizer u^* for J, exists.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \text{ for all } \lambda \in [0,1], \ x, \ y \in \mathbb{R}.$$

(i.) Show that if $a < b < c$, then

$$\frac{f(b) - f(a)}{b - a} \leq \frac{f(c) - f(b)}{c - b}.$$

(ii.) Prove that $f(x)$ is Lipshitz on $[-1,1]$, i.e. there exists $M > 0$ such that

$$\big|f(y) - f(x)\big| \leq M|y - x| \text{ for all } x, y \in [-1,1].$$

6. Set $\mathbb{R}_+ = [0, +\infty)$. It is known that if a function $f : \mathbb{R}_+ \to \mathbb{R}$ is uniformly continuous, and that $f \in L^1(\mathbb{R}_+)$, then

$$\lim_{x \to +\infty} f(x) = 0.$$

Accept this result without proving it.

Give counterexample in case uniform continuity is relaxed to continuity.