Department of Mathematics OSU
Qualifying Examination
Fall 2016

Real Analysis

- Do any of the four of the six problems. Indicate on the selection sheet with your identification number which four you are submitting.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete this exam.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination, place examination blue book(s) and selection sheet back into the envelope in which the test materials came. You will hand in all materials. If you use extra examination books, be sure to place your code number on them.

DO NOT WRITE YOUR NAME ANYWHERE – USE ONLY YOUR TEST ID CODE

Standard Notations: $\mathbb{R}$ denotes the reals, $\mathbb{C}$ denotes the complex numbers, m_n is Lebesgue measure on $\mathbb{R}^n$ (the subscript will be omitted when $n = 1$), if more than one variable appears in an integration problem the variable name may be included dm as, for example, $dm(x)$. For any Hausdorff space M, $\mathcal{B}(M)$ denotes the sigma-algebra of Borel sets (the sigma-algebra generated by the open sets), set complementation is denoted with ^{c}, i.e. A^c will denote the complement of A with respect to the ambient space, but set difference notation $A \setminus B$ will be used for relative complement.

1. (a) Assume that a function $f : [0,1] \to \mathbb{R}$ is Lebesgue integrable on $[0,1]$ and that $\int_E f(x)\, dm = 0$ for every Lebesgue measurable subset E of $[0,1]$. Prove that $f = 0$ almost everywhere in $[0,1]$.

   (b) Now assume that the function f in part (a) is continuous at every point in $[0,1]$. Prove the same result as in part (a), but use a proof that has an "advanced calculus" flavor, i.e., use ϵ, δ, and the definition of continuity.

   (c) Give, and justify, an example of a Lebesgue integrable function $f : [0,1] \to \mathbb{R}$ for which the set $\{x \in [0,1] : f(x) \neq 0\}$ has positive Lebesgue measure but does not contain any interval of positive length.

2. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions on $\mathbb{R}$ that converges *pointwise* everywhere on $\mathbb{R}$. (The convergence need not be uniform.) Let $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in \mathbb{R}$. Prove that there exists a real number K and a nonempty open interval I such that $f(x) \leq K$ for all $x \in I$.

   Suggestion As part of your proof, include the following steps:

   - For each integer $M \geq 1$, let $E_M = \{x \in \mathbb{R} : f_n(x) \leq M \text{ for all } n \geq 1\}$. Prove that E_M is a closed set, for all $M \geq 1$.
   - Prove that $\mathbb{R} = \cup_{M=1}^{\infty} E_M$.

3. Let α and M be real numbers such that $0 < \alpha \leq 1$ and $M > 0$. Consider the Banach space $X = C[0,1]$, i.e. the set of continuous real-valued functions defined on the interval $[0,1]$ with the usual sup norm. We say that a function f is Hölder continuous of order α with Hölder constant M, if for all x, y in $[0,1]$ it holds that:

   $$|f(x) - f(y)| \leq M|x - y|^{\alpha}.$$

   (a) Show that if f is Hölder continuous of order α with Hölder constant M, then f belongs to X.

   (b) Let $C_M^{\alpha}[0,1]$ denote the set of Hölder continuous functions of order α with Hölder constant M. Show that $C_M^{\alpha}[0,1]$ is not compact in X.

   (c) Prove that $C_M^{\alpha}[0,1] \cap \mathcal{B}_1$ is compact in X, where $\mathcal{B}_1$ is the closed unit ball in X.

4. Let f be Lebesgue integrable on $[0, b]$ for some $b > 0$, and define for $0 < x \leq b$,

$$g(x) = \int_x^b \frac{f(t)}{t}\, dm(t).$$

Prove that g is integrable on $[0, b]$ and that

$$\int_0^b g(x)\, dm(x) = \int_0^b f(t)\, dm(t).$$

5. (a) Let $\{f_n\}$ be a sequence of functions in $L^p([a, b])$ for p with $1 \leq p < \infty$ and some bounded interval $[a, b]$ in $\mathbb{R}$. State (no proofs or examples needed) the implications between convergence in L^p, convergence almost everywhere, and convergence in measure for the sequence $\{f_n\}$ or subsequence thereof.

   (b) Let E be a bounded measurable subset of $\mathbb{R}$ with positive measure. Using that

$$\lim_{a \to 0} \int |\chi_E(x) - \chi_E(x - a)|\, dm(x) = 0,$$

   prove that there exists a sequence of positive real numbers $\{h_j\}$ decreasing to zero, so that for almost every $x \in E$, there exists a natural number $N(x)$ so that $x + h_j$ and $x - h_j$ also belong to E when $j \geq N(x)$.

6. We consider a differential equation

$$\frac{dx}{dt} = f(t, x),$$

where $f : [0, T] \times U \to \mathbb{R}^n$ for a given $T > 0$ and open set U in $\mathbb{R}^n$. Let $\| \ \|$ denote the Euclidean norm on $\mathbb{R}^n$ and assume that f satisfies:

- (H1) For all x in U, $f(t, x)$ is (Lebesgue) measurable with respect to t in $[0, T]$.
- (H2) f is bounded by an integrable function $M : [0, T] \to [0, \infty)$:

$$\|f(t, x)\| \leq M(t), \text{ for all } x \text{ in } U \text{ and all } t \text{ in } [0, T].$$

- (H3) f is Lipschitz in x, i.e. there is an integrable function $K : [0, T] \to [0, \infty)$ such that:

$$\|f(t, x) - f(t, y)\| \leq K(t)\|x - y\|, \text{ for all } x, y \text{ in } U \text{ and all } t \text{ in } [0, T].$$

We say that a function $x : [0, T'] \to U$, where T' is positive and $T' \leq T$, is a (locally defined) solution of the differential equation starting at a given x_0 in U, if $x(t)$ is continuous on $[0, T']$, $x(0) = x_0$, and $x(t)$ satisfies the integral equation:

$$x(t) = x_0 + \int_0^t f(s, x(s))ds, \text{ for all } t \text{ in } [0, T'].$$

Prove that for every x_0 in U, there is a solution starting at x_0. [Note: You may assume the result that if $g(\xi, \eta)$ is Lebesgue measurable in ξ for each fixed η and continuous in η for each fixed ξ, and if h is Lebesgue measurable, then $w(\xi) = g(\xi, h(\xi))$ is Lebesgue measurable.]