Department of Mathematics OSU
Qualifying Examination
Spring 2016

Real Analysis

- Do any four of the six problems. Indicate on the sheet with your identification number the four which you wish to have graded.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete this exam.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination, place examination blue book(s) and selection sheet back into the envelope in which the test materials came. You will hand in all materials. If you use extra examination books, be sure to place your code number on them.

DO NOT WRITE YOUR NAME ANYWHERE – USE ONLY YOUR TEST ID CODE

Standard Notations and Conventions: $\mathbb{R}$ denotes the reals, $\mathbb{C}$ denotes the complex numbers, m_n is Lebesgue measure on $\mathbb{R}^n$ (the subscript will be omitted when $n = 1$), for any Hausdorff space M, $\mathcal{B}(M)$ denotes the sigma-algebra of Borel sets (the sigma-algebra generated by the open sets), set complementation is denoted with ^{c}, i.e. A^c will denote the complement of A with respect to the ambient space, but set difference notation $A \setminus B$ will be used for relative complement. For a measurable set E, $1 \le p < \infty$, $L^p(E)$ denotes the set of (equivalence classes of) p-th power integrable functions with norm $\|f\|_p = \left(\int_E |f|^p \, dm\right)^{1/p}$ and for $p = \infty$ denotes the set of (equivalence classes of) essentially bounded measurable functions with essential supremum as norm. Unless otherwise noted, you may assume that functions in L^p are real valued.

1. Let A be a measurable subset of $\mathbb{R}$ with the property that there exists a $\delta > 0$ such that $m(A \cap I) \ge \delta m(I)$ for every interval I of finite measure. Prove that $m(A^c) = 0$.

2. Let $[a, b]$, $a < b$, be a bounded interval in $\mathbb{R}$. Show that $L^2([a, b])$ is a first category subset of $L^1([a, b])$: that is, show that it lies in a countable union of nowhere dense closed subsets. [Hint: Consider the sets $F_k = \{f \in L^2 : \|f\|_2 \le k\}$ for $k \in \mathbb{N}$.]

3. Let I be an interval in $\mathbb{R}$, and let $\{u_n\}_{n=1}^{\infty}$ be a maximal orthonormal subset of $L^2(I)$. Here "orthonomal" means that $(u_m, u_n) = 1$ if $m = n$ and $(u_m, u_n) = 0$ if $m \ne n$, where $(\cdot, \cdot)$ is the inner product on $L^2(I)$. The term "maximal" means that the set $\{u_n\}_{n=1}^{\infty}$ cannot be enlarged to form another orthonormal set; that is, if $(v, u_m) = 0$ for all m, then $v = 0$.

   (a) Let $f \in L^2(I)$, and assume that $f = \sum_{n=1}^{\infty} c_n u_n$, where c_n is a real number, for each n. Here, the series converges in the sense of $L^2(I)$, i.e., if $S_N = \sum_{n=1}^{N} c_n u_n$ for each $N \ge 1$, then $\|f - S_N\|_2 \to 0$ as $N \to \infty$. Prove that $c_n = (f, u_n)$ for all $n \ge 1$.

   Note. Be careful with the infinite sum. Try $(f, u_m) = (f - S_N, u_m) + (S_N, u_m)$.

   (b) Prove that *every* $f \in L^2(I)$, the series $\sum_{n=1}^{\infty} (f, u_n)^2$ converges. *Suggestion.* Begin with $0 \le \|f - \sum_{n=1}^{N} (f, u_n) u_n\|_2^2$.

   (c) Prove that for every $f \in L^2(I)$, the series $\sum_{n=1}^{\infty} (f, u_n) u_n$ converges in $L^2(I)$. (One step is to prove that the sequence of partial sums is a Cauchy sequence.)

   (d) Prove that every $f \in L^2(I)$ can be expressed in the form $f = \sum_{n=1}^{\infty} (f, u_n) u_n$. *Suggestion.* Given $f \in L^2(I)$, define r so that $f = r + \sum_{n=1}^{\infty} (f, u_n) u_n$, and use maximality to prove $r = 0$.

Exam continues on next page ...

4. Prove that if $f \in L^1(\mathbb{R})$, then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x) \cos nx \, dx = 0$$

You may use the fact, without proving it, that the set of all step functions with compact support is dense in $L^1(\mathbb{R})$.

5. There are two parts to this problem. You may use the result of the first part in the second part, even if you do not solve the first part.

   (a) Let $\{a_n\}$ be a sequence of real numbers which is subadditive, i.e.

   $$a_{n+m} \leq a_n + a_m, \text{ for all } n, m \in \mathbb{N}.$$

   Prove that

   $$\lim_{n\to\infty} \frac{a_n}{n} = \inf_n \left(\frac{a_n}{n}\right).$$

   (This limit may be $-\infty$, e.g. if $a_n = -n^2$.)
   [Hint: For any $q, n \in \mathbb{N}$ with $n \geq q$, there exists $m_n \in \mathbb{N}$ and a non-negative integer r_n such that $n = m_n q + r_n$.]

   (b) Let X be a Banach space, let $\{T_n\}$ be a sequence of bounded operators on X, and let $\|T_n\|$ denote the operator norm. Assume that $T_{n+m} = T_n T_m$ (composition) for all n, m. Accepting the convention that $\ln(0) = -\infty$, prove that

   $$\lim_{n\to\infty} \frac{1}{n} \ln(\|T_n\|)$$

   exists as an extended real number, and is less than $+\infty$.

6. Let $a < b$ be real numbers, and $f : \mathbb{R} \times [a, b] \to \mathbb{R}$ be a continuous function. Assume that f is differentiable with respect to its first variable, and that there exist $0 < m < M < \infty$ such that

$$m \leq \frac{\partial f}{\partial x}(x, y) \leq M,$$

for all $(x, y) \in \mathbb{R} \times [a, b]$. Prove that there exists a unique continuous function $w : [a, b] \to \mathbb{R}$ such that $f(w(y), y) = 0$ for all $y \in [a, b]$. [Hint: Show that the operator F which maps the function $z(y)$ for $y \in [a, b]$, to

$$F(z)(y) := z(y) - \frac{1}{M} f(z(y), y)$$

is a contraction on a suitable function space.]