Department of Mathematics OSU
Qualifying Examination
Spring 2015

PART II: COMPLEX ANALYSIS and LINEAR ALGEBRA

- Do any of the two problems in Part CA, *use the correspondingly marked blue book* and indicate on the selection sheet with your identification number those problems that you wish graded. Similarly for Part LA.

- Your solutions should contain all mathematical details. Please write them up as clearly as possible.

- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.

- On problems with multiple parts, individual parts may be weighted differently in grading.

- You have three hours to complete Part II.

- When you are done with the examination, place examination blue books and selection sheets back into the envelope in which the test materials came. You will hand in all materials. If you use extra examination books, be sure to place your code number on them.

PART CA : COMPLEX ANALYSIS QUALIFYING EXAM

1. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $f(\mathbb{R}) \subseteq \mathbb{R}$ and $f(i\mathbb{R}) \subseteq i\mathbb{R}$, where $i\mathbb{R} = \{it \mid t \in \mathbb{R}\}$. Show that $f(-z) = -f(z)$ for all $z \in \mathbb{C}$.

2. Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$, and let $f : D \to D$ be a holomorphic map. Show that for any distinct $z, w \in D$,
$$\left| \frac{f(z) - f(w)}{\overline{f(w)} f(z) - 1} \right| \leq \left| \frac{z - w}{\overline{w} z - 1} \right|.$$

3. Find the number of roots of $z^7 - 4z^3 - 11 = 0$ contained in the annulus $1 < |z| < 2$.

Exam continues on next page ...

PART LA: LINEAR ALGEBRA QUALIFYING EXAM

1. Let $A = (a_{ij})$ be a nonsingular $n \times n$ matrix with entries in $\mathbb{C}$, and let $A^* = (\overline{a}_{ji})$ be its complex conjugate transpose (or adjoint). Let $\|\mathbf{x}\| = (|x_1|^2 + \cdots + |x_n|^2)^{1/2}$ denote the standard norm on $\mathbb{C}^n$.

   (a) Show that all of the eigenvalues of A^*A are positive real numbers.

   (b) Show that for all $\mathbf{x} \in \mathbb{C}^n$,

   $$\sqrt{\lambda_{\min}} \cdot \|\mathbf{x}\| \leq \|A\mathbf{x}\| \leq \sqrt{\lambda_{\max}} \cdot \|\mathbf{x}\|$$

   where $\lambda_{\min}$ is the smallest eigenvalue of A^*A and $\lambda_{\max}$ is the largest eigenvalue of A^*A.

2. Let F be an arbitrary field. Recall that a matrix M is called *nilpotent* if there is a positive integer r for which $M^r = 0$.

   (a) Prove that any Jordan block B of a matrix in Jordan canonical form with entries in F is similar to its own transpose.

   (b) Let N be any $n \times n$ nilpotent matrix with entries in F. Prove that N is similar to its transpose.

3. Let A, B, M be $n \times n$ matrices over $\mathbb{C}$ such that $AM = MB$.

   (a) If $f(x)$ is the minimal polynomial for B, show that $f(A)M = 0$.

   (b) If $M \neq 0$, show that A and B must have a common eigenvalue.