Department of Mathematics OSU
Qualifying Examination
Spring 2015

PART I : Real Analysis

- Do any four of the six problems in Part I. Indicate on the sheet with your identification number the four which you wish graded.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete Part I.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination, place examination blue book and selection sheet into the unmarked, smaller envelope. You will hand in all materials. If you use extra examination books, be sure to place your code number on them.

1. Let $\{f_n\}$ be a sequence of measurable functions on the interval (a, b). Show that $\int_a^b \frac{|f_n|}{1+|f_n|} dx \to 0$ if and only if $f_n \to 0$ in measure. [That is, for all $\varepsilon > 0$, the sets $E_n = \{x \in (a, b) : |f_n(x)| \geq \varepsilon\}$ satisfy, $m(E_n) \to 0$.]

2. Let $(t_0, y_0) \in \mathbb{R} \times \mathbb{R}$ and the function $f(t, y)$ be continuous in the rectangle $R = [t_0 - a, t_0 + a] \times [y_0 - b, y_0 + b]$ in $\mathbb{R} \times \mathbb{R}$, and for some $K > 0$ assume

$$|f(t, y_1) - f(t, y_2)| \leq K|y_1 - y_2| \text{ for all } (t, y_1), (t, y_2) \in R.$$

Show that there is an $h > 0$ such that the initial-value problem

$$y'(t) = f(t, y(t)) \text{ for } |t - t_0| < h, \ y(t_0) = y_0,$$

has a unique solution.

3. Let Γ be the image of a Lipschitz continuous map $\gamma : [0, 1] \to \mathbb{R}^n$, i.e. $\Gamma = \{\gamma(t) : t \in [0, 1]\}$. [That is, $|\gamma(t_1) - \gamma(t_2| \leq K|t_1 - t_2|$ for all $t_1, t_2 \in [0, 1]$.]
Prove that $m(\Gamma)$=0 if $n \geq 2$, where $m(\cdot)$ is Lebesgue measure.

4. Set $\mathbb{R}_+ = [0, +\infty)$. Suppose that $f : \mathbb{R}_+ \to \mathbb{R}$ is uniformly continuous, and that $f \in L^1(\mathbb{R}_+)$. Show that:

$$\lim_{x \to +\infty} f(x) = 0.$$

5. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set of positive Lebesgue measure, i.e. $m(E) > 0$. Prove that there is an interval I such that $m(I) > 0$ and

$$m\Big(E \cap I\Big) \geq \frac{3}{4} m(I)$$

6. Suppose $h : [0, \infty) \to \mathbb{R}$ is a continuous function with compact support. Prove

$$\lim_{\varepsilon \downarrow 0} \int_{\varepsilon}^{\infty} \frac{h(\alpha x) - h(\beta x)}{x} dx = h(0) \ln \frac{\alpha}{\beta}$$

for any $\alpha > 0$ and $\beta > 0$.