Department of Mathematics OSU
Qualifying Examination
Fall 2011

PART II: COMPLEX ANALYSIS and LINEAR ALGEBRA

- Do any of the two problems in Part CX, *use the correspondingly marked blue book* and indicate on the selection sheet with your identification number those problems that you wish graded.

  Similarly for Part LA.

- Your solutions should contain all mathematical details. Please write them up as clearly as possible.

- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.

- You have three hours to complete Part II.

- On problems with multiple parts, individual parts may be weighted differently in grading.

- When you are done with the examination, place examination blue books and selection sheets into the unmarked, smaller envelope. You will hand in all materials. If you use extra examination books, be sure to place your code number on them.

PART CX : COMPLEX ANALYSIS QUALIFYING EXAM

1. (a) Let $f(z)$, $g(z)$ be analytic in some open, connected Ω, and suppose that the function
$$f(z)\overline{f(z)} + g(z)\overline{g(z)}$$
is constant in Ω. Show that both $f(z)$, $g(z)$ must be constant in Ω.

   (b) Let $P(z)$ be a polynomial whose zeros all lie in the upper half plane $\mathbb{H}^+$. Show that the zeros of the derivative of $P'(z)$ must also be contained in $\mathbb{H}^+$.

2. (a) Let $f(z)$ be entire and injective. Show that $f(z) = az + b$, where $a \neq 0$. (Suggestion: You may wish to consider the function $g(z) = f(1/z)$.)

   (b) Let $f(z)$ is analytic in upper half-plane and $|f(z)| < 1$ for all z. Suppose that $f(\mathrm{i}) = 0$. How large can $|f(2\,\mathrm{i})|$ be?

3. (a) Let $f(z)$ be a meromorphic function in the plane with simple poles at $z = 0, 1, 2, \ldots$. Suppose that the residue at each pole is $\mathrm{Res}(f, k) = \dfrac{1}{1+k}$, $k = 0, 1, 2, \ldots$. Let $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ be the Laurent expansion of $f(z)$ in the annulus $2 < |z| < 3$. Find the coefficients a_n for $n < 0$.

   (b) How many zeros (counting multiplicities) does the polynomial $P(z) = z^5 + z^2 - 5z + 1$ have in the annulus $1 < |z| < 2$?

Exam continues on next page ...

PART LA: LINEAR ALGEBRA QUALIFYING EXAM

1. Let A, X, Y be real $n \times n$ matrices such that both X and Y commute with A

   (a) Suppose that the characteristic polynomial of A has distinct roots in $\mathbb{C}$. Prove that X and Y commute with each other.

   (b) Show by example that the hypothesis of (a) is necessary.

2. Let B be a complex $n \times n$ matrix with rank 1.

   (a) What are all the possible Jordan canonical forms for B? Give justification for your answer.

   (b) For each of your forms in part (a), compute the characteristic polynomial of B, and the minimal polynomial of B.

3. Let $\mathcal{M}_n(\mathbb{C})$ denote the vector space of complex $n \times n$ matrices. For B in $\mathcal{M}_n(\mathbb{C})$, denote by $\overline{B^t}$ the conjugate transpose of B, and let $\operatorname{tr}(B)$ denote the trace. The following defines an inner product on $\mathcal{M}_n(\mathbb{C})$:

$$\langle A \,|\, B \rangle = \operatorname{tr}(A\,\overline{B^t}\,) \; .$$

   (a) Let R_D denote the linear operator on $\mathcal{M}_n(\mathbb{C})$ given by right multiplication by D, thus $R_D(A) = AD$. Now, let D be a diagonal element of $\mathcal{M}_n(\mathbb{C})$. Show that R_D is a normal operator.

   (b) Still with diagonal D, express R_D as a linear combination of orthogonal projections.