Department of Mathematics OSU
Qualifying Examination
Fall 2019

Linear Algebra

- Do any of the four of the six problems. Indicate on the sheet with your identification number the four which you wish graded.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete this examination.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination, place examination blue book(s) and selection sheet into the envelope in which the exam came. You will hand in all materials. If you use extra examination books, be sure to place your code number on them.

1. Let F be a field and V be the vector space of $n \times n$ matrices over F.

   (a) Fix $B \in V$ and define a function f_B on V by $f_B(A) = \text{trace}(B^t A)$. Show that f_B is a linear functional on V.

   (b) Show that every linear functional on V is of the above form.

   (c) Show that $B \mapsto f_B$ is an isomorphism of V onto V^*, the dual space of V.

2. Let A be an $m \times n$ real matrix and B an $n \times p$ real matrix. Prove the following.

   (a)
   $$\text{rank}(AB) = \text{rank}(B) - \dim(\, N(A) \cap R(B)\,),$$
   where $N(M)$ is the null space of the matrix M and $R(M)$ is its column space.

   (b)
   $$\text{rank}(AA^t) = \text{rank}(A) = \text{rank}(A^t A).$$

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3. Let A be an $m \times n$ complex matrix and B a $p \times q$ complex matrix. The tensor product of A and B is the $mp \times nq$ matrix given in block form as

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{pmatrix}$$

(a) Let C and D be two complex matrices of sizes $n \times r$ and $q \times s$, respectively. Show that

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

(b) Suppose that A, B are invertible square matrices. Show that

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}.$$

(c) Suppose now that A is an $m \times m$ complex matrix of eigenvalues $\{\lambda_i\}_{i=1}^m$, and that B is a $p \times p$ complex matrix of eigenvalues $\{\mu_j\}_{j=1}^p$. Prove that $A \otimes B$ has eigenvalues $\{\lambda_i \mu_j\}_{1 \le i \le m, 1 \le j \le p}$.

4. Let M be a real $n \times n$ matrix such that each column sum of M is zero (that is, for each j, $\sum_i M_{ij} = 0$).

(a) Show the determinant of M equals zero.

(b) Show that for each column, the n cofactors of elements in that column are all equal. (Recall that the cofactor of element M_{ij} equals $(-1)^{i+j}$ times the minor of element M_{ij}, and the minor of element M_{ij} is the determinant of the matrix obtained from the matrix M by deleting the ith row and jth column).

(c) Show that the vector whose i^{th} entry is the cofactor of any (and hence every) element of the i^{th} column of M gives an element of the nullspace of M.

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5. Let $\mathcal{M}$ be the real vector space of real $n \times n$ matrices. Let $\mathcal{S}$ be the subset of real symmetric matrices, and $\mathcal{A}$ the subset of real skew-symmetric matrices (i.e., $A = -A^T$ whenever A is in $\mathcal{A}$). **Note**: If you cannot solve some part below, you may skip it, and assume the result for use in later parts.

   (a) Show that $\mathcal{S}$ and $\mathcal{A}$ are subspaces of $\mathcal{M}$, and that $\mathcal{M} = \mathcal{S} \oplus \mathcal{A}$.

   (b) Show that the map $< X, Y >:= \mathrm{tr}(XY^T)$ is an inner-product on $\mathcal{M}$.

   (c) Show that $\mathcal{S}$ and $\mathcal{A}$ are orthogonal subspaces of $\mathcal{M}$. Find orthonormal bases for $\mathcal{S}$ and for $\mathcal{A}$. Determine $\dim(\mathcal{S})$ and $\dim(\mathcal{A})$.

   (d) Given X in $\mathcal{M}$, determine a formula to compute the matrix Y in $\mathcal{A}$ which is closest to X with respect to the metric induced by the inner product $<,>$.

6. Let M be a complex $n \times n$ matrix whose eigenvalues $\lambda_1, \ldots, \lambda_n$ satisfy the following relations:

$$\begin{aligned}
\sum_i \lambda_i &= (-1)^1 \\
\sum_{i_1 < i_2} \lambda_{i_1}\lambda_{i_2} &= (-1)^2 \\
&\vdots \\
\sum_{i_1 < i_2 < \cdots < i_{n-1}} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_{n-1}} &= (-1)^{n-1} \\
\lambda_1 \cdots \lambda_n &= (-1)^n
\end{aligned}$$

   (a) Determine the eigenvalues of M.

   (b) Is M diagonalizable?