Department of Mathematics OSU
Qualifying Examination
Fall 2019

Real Analysis

- Do any of the four of the six problems. Indicate on the sheet with your identification number the four which you wish graded.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete this examination.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination, place examination blue book(s) and selection sheet into the envelope in which the exam came. You will hand in all materials. If you use extra examination books, be sure to place your code number on them.

1. Assume $f : [0, \infty) \to \mathbb{R}$ is a continuous function such that $0 \leq f(x) \leq Cx^{-(1+\rho)}$ for all $x > 0$, and for some constants $C, \rho > 0$. Define the sequence of functions $f_k(x) = kf(kx)$.

   (a) Show that for any $r > 0$, $\lim_{k\to\infty} f_k(x) = 0$ uniformly on $[r, \infty)$.

   (b) Show that f_k does not converge uniformly to the function identically 0 on $(0, \infty)$, unless f is identically 0.

2. (a) Given $a \in \mathbb{R}$, denote by $\{a\}$ the fractional part of a; that is,

   $$\{a\} = \min\{a - n : \ n \in \mathbb{Z}, \ n \leq a\}$$

   Suppose that α is an irrational real number. Prove that the set

   $$A_\alpha = \{ \{n\alpha\} : \ n \in \mathbb{Z}\}$$

   is dense in $[0, 1]$.

   (b) A function $f : \mathbb{R} \to \mathbb{R}$ is called *p-periodic* if $f(x + p) = f(x)$ for all $x \in \mathbb{R}$. Prove that any continuous function that is both 1-periodic and α-periodic for some irrational α must be a constant function.

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3. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$. We say f is *anti-continuous* at x_0 if for any $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - x_0| < \delta$, then $|f(x) - f(x_0)| > \epsilon$.

   (a) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ with a non-empty set of anti-continuity points.

   (b) Prove that for any $f : \mathbb{R} \to \mathbb{R}$ the set of all points where f is anti-continuous is nowhere dense. (Recall that a set A is nowhere dense if the closure of A, $\overline{A}$, has no interior points.)

4. Assume $f_n : [0, \infty) \to \mathbb{R}$ is a sequence of continuous functions that converges uniformly to f on $[0, \infty)$.

   (a) Show that for any $b > 0$,

   $$\lim_{n\to\infty} \int_0^b f_n(x)\mathrm{d}x = \int_0^b f(x)\mathrm{d}x.$$

   You can assume without proof that f is continuous and thus Riemann integrable.

   (b) Give an example to illustrate that the assumptions made are insufficient to guarantee that

   $$\lim_{n\to\infty} \int_0^\infty f_n(x)\mathrm{d}x = \int_0^\infty f(x)\mathrm{d}x.$$

   (c) Give a condition on the sequence f_n that is sufficient to imply that

   $$\lim_{n\to\infty} \int_0^\infty f_n(x)\mathrm{d}x = \int_0^\infty f(x)\mathrm{d}x.$$

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5. Let (X, d) be a compact metric space and let $\mathcal{F}$ be the family of all non-empty compact subsets of X. Define the *Hausdorff distance* on $\mathcal{F}$ by

$$D_H(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B} d(a, b),\ \sup_{b\in B}\inf_{a\in A} d(a, b),\right\}$$

   (a) Explain why D_H is a well-defined metric on $\mathcal{F}$.

   (b) For $x \in X$ and $A \in \mathcal{F}$, let $d(x, A) = \inf\{d(x, a) \mid a \in A\}$. For real $\epsilon > 0$ define

$$A_\epsilon = \{x :\ d(x, A) \leq \epsilon\}.$$

   Show $D_H(A, B) \leq \epsilon$ if and only if both $A \subseteq B_\epsilon$ and $B \subseteq A_\epsilon$.

   (c) Show that the metric space $(\mathcal{F}, D_H)$ is complete.

6. Let $\mathcal{C}([0, 1]) = \{f : [0, 1] \to \infty\}$ denote the space of continuous functions. You can use that this is a complete metric space with respect to the norm $\| f \|= \sup_{x\in[0,1]} |f(x)|$.

   Define

$$\mathcal{F} = \{\, f \in \mathcal{C}([0, 1]) : |f(x) - f(y)| \leq |x - y|, \forall x, y \in [0, 1]\,\}.$$

   (a) Show that $\mathcal{F}$ is closed, but not compact in $\mathcal{C}[0, 1]$.

   (b) Show that

$$\mathcal{F}_1 = \{f \in \mathcal{F} : \int_0^1 f^2(x)\mathrm{d}x = 1\}$$

   is compact in $\mathcal{C}[0, 1]$.