Department of Mathematics OSU
Qualifying Examination
Fall 2013

PART I : Real Analysis

- Do any four of the problems in Part I. Indicate on the sheet with your identification number the four which you wish graded.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete Part I.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination, place examination blue book and selection sheet into the unmarked, smaller envelope. You will hand in all materials. If you use extra examination books, be sure to place your code number on them.

1. Suppose $A_j \subset [0,1]$ is Lebesgue measurable with measure $\geq \frac{1}{2}$ for each $j = 1, 2, \ldots$. Prove that there is a measurable set $S \subset [0,1]$ with measure $\geq \frac{1}{2}$ such that each $x \in S$ is in A_j for infinitely many j.

2. Let the function $F : [0,a] \times [y_0 - r, y_0 + r] \to \mathbb{R}$ be such that

   - $t \mapsto F(t, \xi)$ is measurable for each $\xi \in [y_0 - r, y_0 + r]$,
   - $\xi \mapsto F(t, \xi)$ is continuous for each $t \in [0,a]$,
   - and F is bounded: $|F(t,\xi)| \leq m$ for all t, ξ.

   Assume $a \leq \frac{r}{m}$.

   (a) Show that the map $y \mapsto T(y)$ defined by

   $$T(y)(t) = y_0 + \int_0^t F(s, y(s))\, ds$$

   on the metric space

   $$X = \{y \in C[0,a] : \ |y(t) - y_0| \leq r \text{ for all } t \in [0,a]\}$$

   is compact.

   (b) Show that there exists a solution of the initial-value problem

   $$y'(t) = F(t, y(t)), 0 \leq t \leq a, \quad y(0) = y_0.$$

Exam continues on next page ...

3. Let V be a real vector space and $\varphi : V \to \mathbb{R}$ be *directionally differentiable*: for each $u,\ v \in V$ the one-sided limit $\varphi'(u)(v) = \lim_{t\to 0^+}(\varphi(u+tv) - \varphi(u))/t$ exists. (This is the directional derivative of φ at the point u in the direction v.)

   Show the following are equivalent:

   (a) φ is convex, that is:

   $$\varphi(tu + (1-t)v) \le t\varphi(u) + (1-t)\varphi(v)$$

   for all $u,\ v \in V$ and $t \in [0,1]$.

   (b) $\varphi'(u)(v-u) \le \varphi(v) - \varphi(u)$ for all $u,\ v \in V$.

   (c) $(\varphi'(u) - \varphi'(v))(u-v) \ge 0$ for all $u,\ v \in V$.

4. Let $f : \mathbb{R} \to [0,\infty]$ be integrable with Lebesgue measure μ on $\mathbb{R}$, and define $F : [0,\infty) \to [0,\infty]$ by

   $$F(\alpha) = \mu\{x \in \mathbb{R} : f(x) > \alpha\}.$$

   Show that F is decreasing and right-continuous, and that $F(\alpha) \to 0$ as $\alpha \to \infty$.

Exam continues on next page ...

5. Consider a non-negative function $\varphi(x) : \mathbb{R} \to [0, \infty)$ that integrates to one,

$$\int_{-\infty}^{\infty} \varphi(x)\ dx = 1$$

Prove that for any bounded continuous function $f(x) \in L^\infty(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$ and any point $x \in \mathbb{R}$,

$$n \cdot \int_{-\infty}^{\infty} f(y) \cdot \varphi\big(n(x-y)\big)\ dy \ \longrightarrow\ f(x) \qquad \text{as } n \to \infty\ .$$

6. The norm of a linear functional $L : H \to \mathbb{R}$ on a Hilbert space H is defined by

$$||L|| = \sup_{||x||=1} |Lx|\,,$$

where $||x|| = \sqrt{\langle x, x \rangle}$ and $\langle x, x \rangle$ is the inner product in H.

Show that the linear functionals on H of bounded norm form a Banach space.