**OSU Department of Mathematics**
**Qualifying Examination**
**Spring 2022**

**Real Analysis**

**Instructions:**

- Do **any three** of the four problems.
- Use separate sheets of paper for each problem. Clearly indicate the problem and page number (if several pages are used for a solution) on the top of the page.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have **four** hours to complete this examination.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination:
  1. Use the problem selection sheet to indicate your identification number and the three problems which you wish to be graded.
  2. Arrange your solutions according to the problem order with the problem selection sheet on top and any scratch-work on the bottom.
  3. Submit the exam: place your solutions together with the selection sheet and scratch paper, in the order arranged as above, into the envelope in which you received the exam and submit it to the proctor.

**Exam continues on next page ...**

**Common notations:**

- $\|\cdot\|_\infty$ denotes the supremum norm.
- $C[a,b]$ denotes the space of all real-valued continuous functions on $[a,b]$.
- $C^1[a,b]$ denotes the space of all real-valued continuously differentiable functions on $[a,b]$.
- $C_0^1[0,1]$ denotes the space of all functions $f$ in $C^1[0,1]$ such that $f(0)=0$.

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**Problems:**

1. Assume $1 \le p < \infty$, and define

$$\begin{aligned} \|x\|_p &= \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} && \text{for all } x \in \mathbb{R}^n \\ \|f\|_p &= \left(\int_a^b |f(x)|^p dx\right)^{1/p} && \text{for all } f \in C[a,b] \end{aligned}$$

   **a.** (5 pts) Prove that $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$ for all $x \in \mathbb{R}^n$. Here $\|x\|_\infty = \max_{1\le i\le n} |x_i|$.

   **b.** (5 pts) Prove that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$ for all $f \in C[a,b]$. Here $\|f\|_\infty = \max_{a\le x\le b} |f(x)|$.

2. Recall that a function $\phi : [0,1] \to \mathbb{R}$ is said to be a *step function* if it can be represented as

$$\phi(x) = \sum_{j=1}^{n} c_j \mathcal{I}_j(x),$$

   where $n \ge 0$ is a finite integer, $c_j$ are real numbers, and

$$\mathcal{I}_j(x) = \begin{cases} 1 & \text{if } x \in E_j \\ 0 & \text{if } x \notin E_j \end{cases}$$

   are the characteristic functions for the pairwise disjoint intervals $E_j$. The intervals $E_j$ can be either closed, open, or half-open, and pairwise disjoint means that $E_j \cap E_k = \emptyset$ if $j \neq k$.

   **a.** (4 pts) Use properties of continuous functions to prove that for every $f \in C[0,1]$ and every $\epsilon > 0$, there exists a step function $\phi$ such that

$$\max_{0\le x\le 1} |f(x) - \phi(x)| < \epsilon.$$

   **b.** (2 pts) Prove that $\lim_{n\to\infty} \int_0^1 \phi(x)\cos(nx)\,dx = 0$ for every step function $\phi$ on $[0,1]$.

   **c.** (4 pts) Prove that $\lim_{n\to\infty} \int_0^1 f(x)\cos(nx)\,dx = 0$ for every $f \in C[0,1]$.

3. Consider space

$$S = \{f \in C[0,1] \;:\; f(0) = 0 \;\text{ and }\; \|f\|_\infty < 1\}$$

equipped with $\|\cdot\|_\infty$ norm and a transformation $T : C[0,1] \to C[0,1]$ defined by

$$Tf(x) = \frac{1}{4}\int_0^x f^2(y)\,dy + x(1-x), \qquad x \in [0,1].$$

**a.** (2 pts) Show that $T(S) \subset S$, i.e., $T(f) \in S$ for all $f \in S$.

**b.** (4 pts) Show that $T$ is a contraction in space $S$, i.e., $T : S \to S$ is a contraction.

**c.** (4 pts) Prove that there exists a unique differentiable function $f \in S$ satisfying

$$4f'(x) - f^2(x) = 4 - 8x$$

4. (10 pts) Consider a family $\mathcal{F}$ of functions in $C_0^1[0,1]$ with uniformly bounded derivative. That is, there exists $M > 0$ such that $\|f'\|_\infty < M$ for all $f \in \mathcal{F}$. Prove that the family of functions

$$\mathcal{G} = \{f^2(x) \;:\; f \in \mathcal{F}\}$$

is equicontinuous.