OSU Department of Mathematics
Qualifying Examination
Fall 2022

Linear Algebra

Instructions:

- Do **any three** of the four problems.
- Use separate sheets of paper for each problem. Clearly indicate the problem and page number (if several pages are used for a solution) on the top of the page.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have **four** hours to complete this examination.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination:
  1. Use the problem selection sheet to indicate your identification number and the three problems which you wish to be graded.
  2. Arrange your solutions according to the problem order with the problem selection sheet on top and any scratch-work on the bottom.
  3. Submit the exam: place your solutions together with the selection sheet and scratch paper, in the order arranged as above, into the envelope in which you received the exam and submit it to the proctor.

Exam continues on next page ...

Problems:

1. Consider space $\mathbb{R}^2$ equipped with the Euclidean metric $d(\mathbf{x},\mathbf{y}) = \|\mathbf{y}-\mathbf{x}\|$. Let T be an isometry of $\mathbb{R}^2$ onto itself, i.e., a map such that

$$d\big(T(\mathbf{x}), T(\mathbf{y})\big) = d(\mathbf{x},\mathbf{y}) \qquad \forall \mathbf{x},\mathbf{y} \in \mathbb{R}^2.$$

   i. (6 pts) Prove that T is an *affine transformation*, i.e., T can be represented as $T(\mathbf{x}) = \mathbf{a} + A\mathbf{x}$, where $\mathbf{a} \in \mathbb{R}^2$ and A is a real 2×2 matrix.
   Hint: You need to show that $S(\mathbf{x}) = T(\mathbf{x}) - T(\mathbf{0})$ is linear. For this recall that if $d(\mathbf{x},\mathbf{z}) = d(\mathbf{x},\mathbf{y}) + d(\mathbf{y},\mathbf{z})$, then $\mathbf{x},\mathbf{y},\mathbf{z}$ lie on a line with $\mathbf{y}$ between $\mathbf{x}$ and $\mathbf{z}$.

   ii. (4 pts) Prove that matrix A (in part **i** of the problem) is an orthogonal matrix.

2. For $c \in \mathbb{R}$, define the real 3×3 matrix A by

$$A = \begin{pmatrix} c & c-1 & 0 \\ c+1 & c & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

   a. (5 pts) Determine for what values of $c \in \mathbb{R}$ matrix A is diagonalizable over $\mathbb{R}$ and determine the eigenvalues of A in these cases.

   b. (5 pts) Determine for what values of $c \in \mathbb{R}$ matrix A is not diagonalizable over $\mathbb{R}$ but has a real Jordan Canonical Form. Determine the Jordan Canonical Form of A in these cases.

3. Let V be a finite-dimensional complex inner product space. Let T be a skew-Hermition linear operator, i.e., $T^* = -T$.

   a. (5 pts) Show that all eigenvalues of T are purely imaginary.

   b. (5 pts) Show that eigenvectors associated to distinct eigenvalues of T are orthogonal.

4. Consider matrices

$$m = \begin{pmatrix} p+a & 3c \\ 2c & q+2b \end{pmatrix} \qquad \text{and} \qquad M = \begin{pmatrix} p & a & c & c & c \\ a & p & c & c & c \\ c & c & q & b & b \\ c & c & b & q & b \\ c & c & b & b & q \end{pmatrix},$$

   where $a, b, c, p, q \in \mathbb{R}$ and $c \neq 0$. Let λ and μ be the eigenvalues of matrix m with corresponding real eigenvectors $\mathbf{u} = (u_1, u_2)^T$ and $\mathbf{v} = (v_1, v_2)^T$ respectively.

   a. (1 pts) Explain why $\lambda, \mu \in \mathbb{R}$.

   b. (6 pts) Prove that λ and μ are the eigenvalues of M, and determine corresponding eigenvectors.

   c. (3 pts) Find all remaining eigenvalues of M.

   Note: You do not need to determine λ, μ, $\mathbf{u}$, or $\mathbf{v}$.