OSU Department of Mathematics
Qualifying Examination
Spring 2023

Linear Algebra

Instructions:

- Do **any three** of the four problems.
- Use separate sheets of paper for each problem. Clearly indicate the problem and page number (if several pages are used for a solution) on the top of the page.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have **four** hours to complete this examination.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination:
  1. Use the problem selection sheet to indicate your identification number and the three problems which you wish to be graded.
  2. Arrange your solutions according to the problem order with the problem selection sheet on top and any scratch-work on the bottom.
  3. Submit the exam: place your solutions together with the selection sheet and scratch paper, in the order arranged as above, into the envelope in which you received the exam and submit it to the proctor.

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Common notations:

- A^t denotes the transpose of a matrix A
- A^n denotes the n^{th} power of a square matrix A
- $M = \big(m_{ij}\big)$ is the matrix whose (i, j) entry is m_{ij}
- $\det(M)$ denotes the determinant of a square matrix M
- $\mathbb{C}^{d\times d}$ is the space of all $d \times d$ square matrices with coefficients in $\mathbb{C}$
- $\mathbb{C}[x]$ is the space of all polynomials in x with coefficients in $\mathbb{C}$

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Problems:

1. Let $n = 2k+1$ where k is a positive integer. Let W denote the set of polynomials in $\mathbb{C}[x]$ of degree at most n having only even powers of x, i.e.,

$$W = \{a_n x^n + \ldots + a_1 x + a_0 \mid a_i \in \mathbb{C} \text{ and } a_i = 0 \text{ if } i \text{ is odd}\}.$$

   a. (4 pts) Prove that W is a vector subspace of the complex vector space $\mathbb{C}[x]$ and determine a basis for W.

   b. (6 pts) Suppose A is a complex $k \times n$ matrix such that for any $\vec{b} \in \mathbb{C}^k$, the equation $A\vec{x} = \vec{b}$ has a solution. Prove that the null space (or kernel) of A is isomorphic to W as complex vector spaces.

2. Let A and B be two d-dimensional complex square matrices (i.e., A, $B \in \mathbb{C}^{d\times d}$) and suppose $AB = BA$. Prove the following statements.

   a. (5 pts) Prove that A and B have a common eigenvector in $\mathbb{C}^d$ by first showing that if λ is an eigenvalue for A with corresponding eigenspace E_λ, then E_λ is invariant under B (i.e., for any $\mathbf{x} \in E_\lambda$, vector $B\mathbf{x} \in E_\lambda$).

   b. (5 pts) Suppose A and B are Hermitian (self-adjoint), then there is an orthogonal basis of $\mathbb{C}^d$ relative to which both A and B are diagonal matrices.

3. Let $A = \big(a_{i,j}\big)$ be a d-dimensional square matrix with all **integer** entries, i.e., $a_{i,j} \in \mathbb{Z}$ for all i, j.

   a. (5 pts) Prove that if $r \in \mathbb{Z}$ is an eigenvalue of A, then $\det(A)$ is divisible by r in $\mathbb{Z}$.

   b. (5 pts) Use part (a) to prove that if there is an integer $r \in \mathbb{Z}$ such that

$$\sum_{j=1}^{d} j a_{i,j} = ir \qquad \text{for all} \quad i \in \{1, 2, ..., d\},$$

   then, $\det(A)$ is divisible by r.

4. Let $\mathbb{F}_3$ denote the finite field with three elements $\{0, 1, 2\}$, where addition and multiplication is defined modulo 3.

   Define the 3×3 matrix A with entries in $\mathbb{F}_3$ by

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix}.$$

   a. (4 pts) Determine which $\alpha \in \mathbb{F}_3$, if any, are eigenvalues of A and with what multiplicity.

   b. (6 pts) Determine whether A has a Jordan Canonical Form over $\mathbb{F}_3$, and if so, determine what it is.