New Inequalities for the Weil height

Let \( h : \mathbb{Q}^\times \to [0, \infty) \) denote the absolute logarithmic Weil height defined on the multiplicative group \( \mathbb{Q}^\times \) of nonzero algebraic numbers. Because the height of a root of unity is 0, it can be shown that the height is well defined on cosets of the quotient group
\[
\mathcal{G} = \mathbb{Q}^\times / \text{Tor}(\mathbb{Q}^\times).
\]
Then it turns out that \((\alpha, \beta) \mapsto h(\alpha \beta^{-1})\) defines a metric on \(\mathcal{G}\). Recently, the completion of this metric space was discovered to be a real Banach space. This leads to some new inequalities for heights proved using methods from functional analysis. The following is an example (obtained jointly with R. Grizzard):

**Theorem.** Let \( K \subseteq \mathbb{Q} \) be an intermediate field, and let \( \alpha \) be an element of \( \mathbb{Q} \). Assume that for every \( \varepsilon > 0 \), there exists an integer \( m \neq 0 \), and a point \( \beta \) in \( K^\times \), such that
\[
h(\alpha^m \beta^{-1}) < \varepsilon |m|.
\]
Then there exists an integer \( n \neq 0 \) such that \( \alpha^n \) belongs to \( K^\times \).

We will outline the proof of this and other related inequalities for the Weil height.