**Department of Mathematics OSU**
**Qualifying Examination**
**Fall 2009**

## PART II : Linear Algebra and Complex Analysis

- Do any two of the three problems in each section of Part II. Indicate on the sheet with your identification number the four problems which you wish to be graded.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete Part II.
- On problems with multiple parts, individual parts may be weighted differently in grading.

## Linear Algebra Problems

1. Let $V$ be a finite-dimensional vector space over a field $F$. A non-zero linear operator $T$ on $V$ is a called a *projection* if there exist subspaces $W_1, W_2$ such that $V = W_1 \oplus W_2$ and $T(w_1 + w_2) = w_1$ for all $w_i \in W_i$.
   The *trace* of any matrix is the sum of its diagonal entries. For this problem you may assume the standard result that trace is invariant under similarity.
   Prove: If $T$ is a projection on a finite-dimensional vector space then the trace of any matrix representation of $T$ equals the rank of $T$.

2. Let $V$ be a finite-dimensional (Hermitian) inner product space over $\mathbb{C}$ of dimension $2n$. Let $W$ be an $n$-dimensional subspace of $V$, and $W^\perp$ be its orthogonal complement. Let $\{a_1, \ldots, a_n\}$, $\{b_1, \ldots, b_n\}$ be orthonormal bases for $W$, $W^\perp$, respectively. Consider the linear operator $T$ defined by $T(a_i) = b_i$ , $T(b_i) = a_i$ for all $i = 1, \ldots, n$.
   (a) Find the Jordan Canonical Form for $T$.
   (b) Find the orthogonal complements of all eigenspaces of $T$.

3. (a) Let $T$ be a linear operator on the complex space $\mathbb{C}^n$.
   Prove: If $\ker(T - \alpha I)^n = \ker(T - \alpha I)$ for all $\alpha \in \mathbb{C}$ then $T$ is diagonalizable.
   (b) For $A = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$, find all real constants $c$ such that the (entrywise) limit $\lim_{k\to\infty} (cA)^k$ exists and is nonzero.

**Complex Analysis Problems**

1. (a) Find all entire functions $f(z)$ such that $f(x) = \cos x$ for all $x \in \mathbb{R}$.

   (b) Construct analytic functions $f(z),\ g(z) : \mathcal{D}_1 \to \mathcal{D}_1$, where $\mathcal{D}_1$ is the open unit disk centered at the origin, with $f(1/2) = 3/4$, $f'(1/2) = 7/12$, and $g(1/2) = 3/4$, $g'(1/2) = 3/4$, or show that such a function does not exist. Discuss the uniqueness of $f(z)$ and $g(z)$ (provided they exist).

2. (a) Let $f$ be a continuous complex-valued function defined on an open, connected set $\Omega \subset \mathbb{C}$ such that the (complex) integral $\int_\gamma f(z)\,dz = 0$ for all closed piecewise smooth curves $\gamma$ in $\Omega$. Show that $f$ is analytic in $\Omega$. (Note: you may use the fact that the derivative of an analytic function is analytic without proof.)

   (b) Let

   $$f(z) = \int_0^1 \frac{\exp(tz)}{\sin\sqrt{t}}\,dt, \quad z \in \mathbb{C}.$$

   Show that $f$ is entire.

3. Use the calculus of residues to compute the following integrals:

   (a)

   $$\frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{\sin z}{z^4}\,dz,$$

   where $\gamma$ is the unit circle traced in the counterclockwise direction.

   (b)

   $$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx.$$