Department of Mathematics
Qualifying Examination Fall Term 1998

Part I (Real Analysis):

Do any four of the problems in Part I. Your solutions should consist of clear complete explanations that include all essential mathematical details. You have three hours to complete Part I of the exam.

1. Let $\{f_n\}$ be a sequence of continuous real-valued functions on $[0,1]$. Are the following statements true? If so, give a proof. If not, give a counterexample.
(a) If $f_n \to 0$ in the L^1 norm on $[0,1]$, then $f_n \to 0$ in the L^2 norm on $[0,1]$.
(b) If $f_n \to 0$ in the L^2 norm on $[0,1]$, then $f_n \to 0$ in the L^1 norm on $[0,1]$.

2. The norm of a linear functional $L : H \to \mathbb{R}$ on a Hilbert space H is defined by

$$\|L\| = \sup_{\|x\|=1} |Lx|$$

where $\|x\| = \sqrt{\langle x, x\rangle}$ and $\langle x, x\rangle$ is the inner product in H. Establish the following equivalent formulas for the norm of L:

$$\|L\| = \sup_{\|x\|\leq 1} |Lx| = \sup_{x\neq 0} \frac{|Lx|}{\|x\|} = \inf \{M : |Lx| \leq M \|x\| \text{ for all } x\}.$$

3. Let (X, d) and (Y, e) be metric spaces with (Y, e) complete.
(a) Prove: If $A \subseteq X$ then any uniformly continuous function $f : A \to Y$ can be extended to a continuous function $g : \overline{A} \to Y$ on the closure of A in X, and that the extension g is uniformly continuous.
(b) Provide an example to show that uniform continuity in needed in part (a) by exhibiting specific metric spaces (X, d) and (Y, e), a subset A of X, and a continuous function $f : A \to Y$ such that the continuous function does not have a continuous extension to the closure of A into Y.

4. (a) State the Banach Contraction Mapping Fixed Point Theorem.
(b) A mathematical model for the transverse equilibrium displacement $y = y(x)$ of an elastic string with ends pinned is given by (assume this)

$$y(x) = \int_0^1 G(x,s) f(s, y(s))\, ds \qquad (1)$$

where $G(x,s)$, the so-called Green's function for the problem, satisfies $G(x,s) \geq 0$ for $0 \leq x, s \leq 1$, $\int_0^1 G(x,s)\,ds = x(1-x)/2$ for $0 \leq x \leq 1$, and $f(x,y)$ is a continuous forcing term. Use the contraction mapping fixed point theorem to prove that (1) has a unique continuous solution $y = y(x)$ if f satisfies the Lipschitz condition $|f(x,y) - f(x,z)| \leq L|y-z|$ with $L < 8$. Be sure to explain carefully why all the hypotheses in the fixed point theorem are satisfied. What operator do you use? On what space does it act?

5. Let f be a real-valued function in $L^1(-\infty,\infty)$. The Fourier transform of f is the function

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f(x)\,dx$$

defined for all real ω for which the integral exists and is finite.
(a) What is the domain of $\widehat{f}$? That is, for which real ω does the foregoing integral exist and have a finite value.
(b) Prove or disprove: $\widehat{f}$ is continuous on its domain.
(c) If $f(x) = 1$ for $x \in [a,b]$ and $f(x) = 0$ otherwise, show that $\widehat{f}(\omega) \to 0$ as $\omega \to \infty$.
(d) Establish the conclusion in (c) for any f in $L^1(-\infty,\infty)$. *Hint.* Step functions are dense in $L^1(-\infty,\infty)$.

6. Let $\mathcal{F}$ be a set of differentiable functions $f : [0,1] \to \mathbb{R}$ with $f(0) = 0$ and whose derivatives are uniformly bounded in the sense that there is a constant M such that $|f'(x)| \leq M$ for all f in $\mathcal{F}$ and all x in $(0,1)$.
(a) Show that $\mathcal{F}$ is an equicontinuous family of functions.
(b) Show that $\mathcal{F}$ has compact closure in the space $C([0,1],\mathbb{R})$ of continuous real-valued functions on $[0,1]$ with the sup metric (sup norm). Explain briefly.

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Part II (Complex Analysis and Linear Algebra):

A complete exam consists of four problems with two problems chosen from Part CA and two problems chosen from Part LA. Your solutions should consist of clear complete explanations that include all essential mathematical details. You have three hours to complete Part II of the exam.

Part CA

1. Use the residue theorem to evaluate the integral:
$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2}\, dx \text{ where } a > 0.$$

2. Let $f(z)$ be analytic function in the entire complex plane. Suppose there exist real numbers M and $\alpha \geq 0$ such that $|f(z)| \leq M\,(1 + |z|)^{\alpha}$. Prove that $f(z)$ is a polynomial of degree less than or equal to α.

3. Let $f(z)$ and $g(z)$ be analytic functions in the open disc $|z| < 2$. Assume (a) $|f(z)| \geq |g(z)|$ for any z with $|z| = 1$ and (b) $f(z)$ is not zero for any z with $|z| < 1$. Prove that $|f(z)| \geq |g(z)|$ for any z with $|z| < 1$. Give an example which shows that this conclusion is not true without assumption (b).

Part LA

1. Let V be a vector space and $T : V \to V$ a linear transformation.
(a) Suppose V is finite dimensional. Prove that T is one-to-one if and only if T is onto.
(b) Suppose V is infinite dimensional and T is one-to-one. Can we conclude T is onto? Give a proof or a counterexample.
(c) Suppose V is infinite dimensional and T is onto. Can we conclude T is one-to-one? Give a proof or a counterexample.

2. Let A be a 2×2 matrix with real entries such that $A^2 - A + (1/2)I = O$, where I is the 2×2 identity matrix and O is the 2×2 zero matrix. Prove that $A^n \to O$ as $n \to \infty$.

3. Let A and B be $n \times n$ complex matrices such that $AB = BA$.
(a) Let λ be an eigenvalue of A and let W be the λ eigenspace of A, that is, the space of vectors w such that $Aw = \lambda w$. Show that $B(W) \subset W$.
(b) Show that A and B have a common eigenvector.