Department of Mathematics
Qualifying Examination
Fall 2001

Part I: Complex Analysis and Linear Algebra

- Do any two problems in Part CA and any two problems in Part LA.
- Your solutions should include all essential mathematical details; please write them up as clearly as possible.
- State explicitly any standard theorems that are needed to justify your reasoning.
- You have three hours to complete Part I of the exam.
- In problems with multiple parts, the individual parts may be weighted differently in grading.

Part CA

1. Find the Laurent expansion $\sum_{n=-\infty}^{\infty} a_n z_n$ of the function

$$f(z) = \frac{1}{(z-1)(z-2)}$$

   (a) in the region $1 < |z| < 2$;

   (b) in the region $|z| > 2$.

2. Use the Cauchy Integral Formula to prove that if a function $f(z)$ is analytic in a domain D and if z_0 is a point of D, then $f(z)$ has a power series expansion in some open disk centered at z_0. (Do not appeal directly to Taylor's theorem or Laurent's theorem.) State explicitly any other standard theorems that are needed to justify your solution.

3. Let $f(z)$ be an analytic function in an open region of $\mathbf{C}$ containing the closed unit disc D. Suppose $f(0) = 1$ and $|f(z)| > 1$ whenever $|z| = 1$. Show that $f(z)$ has a zero in D.

Part LA

1. Suppose $A = \begin{bmatrix} 0 & 2 \\ -3 & 5 \end{bmatrix}$. Find $A^{10,000}$. Justify your answer.

2. Let V be an n-dimensional complex inner product space, T be a linear operator on V, W be a T-invariant subspace of V with $\dim W = m$, and T^* be the adjoint of T.

   (a) Give an example to show that W need not be T^*-invariant. Verify that your example is T-invariant but not T^*-invariant.

   (b) Assume that W is both T and T^*-invariant. Show there exists a basis β for V such that the matrices of T and T^* with respect to β, $[T]_\beta$ and $[T^*]_\beta$, have the forms

   $$[T]_\beta = \begin{bmatrix} A & O \\ O & B \end{bmatrix} \qquad \text{and} \qquad [T^*]_\beta = \begin{bmatrix} C & O \\ O & D \end{bmatrix}$$

   where A and C are $m \times m$ matrices and B and D are $(n-m) \times (n-m)$ matrices.

3. Suppose $n \times n$ complex matrices A and B have the same characteristic polynomial $p(x)$ and the same minimal polynomial $q(x)$. (Of course, $p(x)$ may not be equal to $q(x)$.) Can we conclude that A and B are similar (i.e., $A = CBC^{-1}$for some invertible $n \times n$-matrix C),

   (a) if $n = 3$?

   (b) if $n = 4$?

In each part give a proof or a counterexample.

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Part II: Real Analysis

- Do any four of the problems in Part II.
- Your solutions should include all essential mathematical details; please write them up as clearly as possible.
- State explicitly any standard theorems that are needed to justify your reasoning.
- You have three hours to complete Part II of the exam.
- In problems with multiple parts, the individual parts may be weighted differently in grading.

1. Let $f : X \to X$ be a map from a metric space into itself. A point $z \in X$ is a *fixed point of* f if $f(z) = z$. Let $\varepsilon > 0$. A point $w \in X$ is an ε-*fixed point of* f if $d(f(w), w) < \varepsilon$.

   (a) Prove: If X is a compact metric space, $f : X \to X$ is a continuous function, and if for every $\varepsilon > 0$ f has an ε-fixed point, then f has a fixed point.

   (b) Prove the following statement or give a counter example: If X is a metric space, $f : X \to X$ is a continuous function, and if for every $\varepsilon > 0$, f has an ε-fixed point, then f has a fixed point.

2. Let $f \in L^p(\mathbf{R})$ for some $1 \leq p < \infty$.

   (a) Show that
   $$\lim_{x\to\infty} \int_x^{x+1} f(t)\,dt = 0\,.$$

   (b) Show, by way of example, that the assertion of part (a) may fail if $p = \infty$.

3. Two norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ on a vector space V are equivalent if there are positive constants m and M such that

$$m \|x\|_\alpha \leq \|x\|_\beta \leq M \|x\|_\alpha$$

for all $x \in V$.

(a) Prove that any two norms on $\mathbb{R}^n$ are equivalent. *Hint.* For any norm $\|\cdot\|$ on $\mathbb{R}^n$ consider the function $f(x) = \|x\|$ on the set

$$\left\{ x = (x_1, ..., x_n) : \sum_{i=1}^{n} |x_i| = 1 \right\}.$$

(b) Show that the following norms on $C[0,1]$, the continuous real-valued functions on $[0,1]$, are not equivalent:

$$\|f\| = \max_{[0,1]} |f(x)| \qquad \text{and} \qquad \|f\|_1 = \int_0^1 |f(x)| \, dx$$

4. Let f be an L^1-function on $[0, \infty)$.

(a) Show that if f is uniformly continuous on $[0, \infty)$ then

$$\lim_{t \to \infty} f(t) = 0.$$

(b) Show, by way of example, that the conclusion of part (a) may fail if f is assumed to be continuous (and L^1) but not uniformly continuous on $[0, \infty)$.

5. Let $f\colon \mathbf{R} \longrightarrow \mathbf{R}$ be a non-negative L^{p_0} function, where $0 < p_0 < \infty$. Show that

$$\lim_{p \to 0^+} \int_R f^p \, d\nu = \nu\left(\{x \in X : f(x) \neq 0\}\right)$$

where ν is the usual Lebesgue measure on the real line.
Hint: Write

$$\int_{\mathbf{R}} f^p d\nu = \int_{X_0} f^p d\nu + \int_{X_1} f^p d\nu + \int_{X_2} f^p d\nu \,,$$

where $X_0 = \{x \in \mathbf{R} : f(x) = 0\}$, $X_1 = \{x \in \mathbf{R} : 0 < f(x) < 1\}$, and $X_2 = \{x \in \mathbf{R} :\ f(x) \geq 1\}$.

6. Let V be the inner product space of all continuous real-valued functions on $[-1, 1]$ with the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\, dt \,.$$

Let W be the subspace of V consisting of odd functions, i.e., $h \in V$ lies in W if and only if $h(-x) = -h(x)$. Find the orthogonal complement $W^\perp$ of W. Justify your answer.