Department of Mathematics
Qualifying Examination
Fall 2004

Part I: Complex Analysis and Linear Algebra

- Do any two problems in Part CA and any two problems in Part LA.
- Your solutions should include all essential mathematical details; please write them up as clearly as possible.
- State explicitly including all hypotheses any standard theorems that are needed to justify your reasoning.
- You have three hours to complete Part I of the exam.
- In problems with multiple parts, the individual parts may be weighted differently in grading.

Part CA

1. Let $z \in \mathbb{C}\backslash \{0\}$. Do the following:

   (a) *Define* what is means for $w \in \mathbb{C}$ to be a (complex) logarithm of z. (Thoughout logarithm means logarithm with base e.)

   (b) Find all logarithms of z; that is, find a formula that expresses all logarithms of z in terms of z.

   (c) Let $R = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$ and define $f : R \to \mathbb{C}$ by

   $$f(z) = \int_1^z \frac{1}{\zeta}\, d\zeta$$

   where the line integral is evaluated along the line segment joining 1 to z. *Prove,* starting with the definition of a derivative, that f is diffentiable and find its derivative. (*Hint.* You may use the Goursat Theorem: If g is analytic in a domain that contains a triangle Δ and its interior, then $\int_\Delta g(\zeta)\, d\zeta = 0$.)

(d) Prove that $f(z)$ is a branch of the logarithm in R; recall that $f(z)$ is a branch of the logarithm in a domain D if $f(z)$ is analytic in D and for each $z \in D$, $f(z)$ is a value of the logarithm. (*Hint.* If this is true, what expression divided by z must be contant?)

2. A point p is a fixed point of a function f if $f(p) = p$. A function f is holomorphic (analytic) on a set S if it is holomorphic in an open set that contains S. Let $D = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane. Do the following.

   (a) Let $f : \bar{D} \to D$ be holomorphic on the closed unit disk. Prove f has a unique fixed point in $\bar{D}$.

   (b) Let $f : D \to D$ be holomorphic on the open unit disk. Prove: If $f(0) = 0$, then

      i. $|f(z)| \leq |z|$ for $z \in D$, and

      ii. Either 0 is the only fixed point of f in D or all points in D are fixed points of f.

3. Either find (with proof) all functions $f(z)$ analytic in $|z| > 0$ and such that $|f(z)| \geq 1/\sqrt{|z|}$ in $|z| > 0$ or prove that no such function exists.

Part LA

1. Fix a positive integer n and let $\mathcal{P}$ be the vector space of all polynomials of degree n or less over the reals. Define a linear transformation $T : \mathcal{P} \to \mathcal{P}$ by $Tp(x) = xp'(x)$ where $p'(x)$ is the derivative of the polynomial p. Do the following, providing convincing justification for your answers.

   (a) Find the kernel (null space) of T.

   (b) Find the range of T. (This means give a simple description of the polynomials that make up the range of T. The description "All polynomials of the form Tp for p in $\mathcal{P}$" is not allowed.)

   (c) Determine all the eigenvalues and eigenvectors of T.

   (d) Find the Jordan canonical form (of a matrix representation) of T.

2. Let A be a complex square matrix and assume that $A^m = I$ where m is a positive integer.

   (a) Show that if λ is an eigenvalue of A, then $\lambda^m = 1$.

   (b) Prove that A is diagonalizable.

3. Let A and B be $n \times n$ nonsingular complex matrices and suppose that $ABA = B$.

   (a) Prove that if v is an eigenvector of A, then so is Bv.

   (b) Prove that A and B^2 have a common eigenvector.

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Part II: Real Analysis

- Do any four of the problems in Part II.
- Your solutions should include all essential mathematical details; please write them up as clearly as possible.
- State explicitly including all hypotheses any standard theorems that are needed to justify your reasoning.
- You have three hours to complete Part II of the exam.
- In problems with multiple parts, the individual parts may be weighted differently in grading.

1. Let X be a metric space with metric d. Suppose every infinite subset of X has a limit point. Prove that X has a countable dense set.

2. We define a real-valued function on $\mathbb{R}^2$ to be locally varying if for each non-empty open set $U \subseteq \mathbb{R}^2$ the function is not constant on U. Show that $\mathbb{R}^2$ can not be written in the form

$$\mathbb{R}^2 = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \{\ x \ : f_i(x) = c_j\ \}$$

where each $f_i : \mathbb{R}^2 \to \mathbb{R}$ is continuous and locally varying and each c_j is a real number.

3. Let λ be Lebesgue measure on $\mathbb{R}^n$. Assume $f_1, f_2, \cdots$ are nonnegative functions in $L^1(\mathbb{R}^n)$, that

$$\lim_{k\to\infty} f_k(x) = f(x) \ \text{ exists a.e. in } \ \mathbb{R}^n,$$

and that $f \in L^1(\mathbb{R}^n)$. If

$$\int f\, d\lambda = \lim_{k\to\infty} \int f_k\, d\lambda$$

show that

(a)

$$\lim_{k\to\infty} \int |f_k - f| \, d\lambda = 0.$$

(b) For every measurable set E,

$$\int_E f \, d\lambda = \lim_{k\to\infty} \int_E f_k \, d\lambda.$$

4. You may assume the following: $L^2[-\pi,\ \pi]$ is a real Hilbert space with inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx,$$

and the set

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\sin nx}{\sqrt{\pi}}, \frac{\cos nx}{\sqrt{\pi}} : n = 1,\ 2,\ 3,\ \cdots \right\}$$

is an orthogonal basis for $L^2[-\pi,\ \pi]$. Let V be the vector subspace spanned by the set

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}} \right\}.$$

(a) Find a function $g \in V$ such that

$$\int_{-\pi}^{\pi} xf(x)\,dx = \int_{-\pi}^{\pi} g(x)f(x)\,dx \qquad \text{for all} \quad f \in V$$

and show there is only one $g \in V$ that satisfies the foregoing condition.

(b) Find all possible solutions $g \in L^2[-\pi,\ \pi]$ that satisfy the equation in (a).

5. Do the following:

(a) Assume $\{f_n\}_{n=1}^{\infty}$ is a sequence of Borel measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$. Show $\limsup_{n\to\infty} f_n$ is a Borel measurable function.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Prove that the derivative f' is Borel measurable.

6. Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set and $f : E \to [0, \infty)$ be a Lebesgue measurable function. Suppose

$$A = \{(x, y) \in \mathbb{R}^{n+1} : 0 \leq y \leq f(x),\ x \in E\}.$$

Let λ_1 denote the Lebesgue measure in $\mathbb{R}^1$, λ_n denote the Lebesgue measure in $\mathbb{R}^n$, and λ_{n+1} denote the Lebesgue measure in $\mathbb{R}^{n+1}$.

(a) Show that the set A is Lebesgue measurable on $\mathbb{R}^{n+1}$.

(b) Show that

$$\lambda_{n+1}(A) = \int_E f(x)\, d\lambda_n(x) = \int_0^\infty \lambda_n\left(\{x \in E : f(x) \geq y\}\right)\, d\lambda_1(y).$$