

Department of Mathematics OSU
Qualifying Examination
Spring 2019

Linear Algebra

- Do any of the four of the six problems. Indicate on the sheet with your identification number the four which you wish graded.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete this examination.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination, place examination blue book(s) and selection sheet into the envelope in which the exam came. You will hand in all materials. If you use extra examination books, be sure to place your code number on them.

1. Let A be a complex $n \times n$ matrix and b be a complex $n \times 1$ vector.
- a.) Suppose that y is a nonzero complex $1 \times n$ vector. Show that there is a nonzero polynomial $q(x)$ and an eigenvalue λ of A such that $yq(A)$ is nonzero, and $(yq(A))A = \lambda(yq(A))$.

b.) Prove that

$$\text{Rank}[b \quad Ab \quad A^2b \quad \cdots \quad A^{n-1}b] = n$$

if and only if

$$\text{Rank}[A - \lambda I \quad b] = n, \text{ for every eigenvalue } \lambda \text{ of } A.$$

Here, the matrix I denotes the $n \times n$ identity matrix. Notice that the first matrix of the statement to be proven is an $n \times n$ matrix, and the second one is an $n \times (n + 1)$ matrix.

2. Let A and B be nonsimilar $n \times n$ complex matrices with both the same minimal polynomial and the same characteristic polynomial. Show that $n \geq 4$ and that the common minimal polynomial does not equal the common characteristic polynomial.

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3. Let $A(t)$ be a real-valued, continuous $n \times n$ matrix function, defined for all $t \in \mathbb{R}$, and $\Phi(t)$ be a real-valued, continuously differentiable $n \times n$ matrix function, such that

$$\Phi'(t) = A(t) \Phi(t), \text{ for all } t \in \mathbb{R}.$$

Here, $\Phi'(t)$ is the matrix function obtained from $\Phi(t)$ by taking the derivatives of each of its entries. Show that

$$\frac{d}{dt} (\det(\Phi(t))) = \text{tr}(A(t)) \det(\Phi(t)), \text{ for all } t \in \mathbb{R},$$

where $\det(\Phi(t))$ denotes the determinant of $\Phi(t)$, and $\text{tr}(A(t))$ denotes the trace of $A(t)$.

You may use that for any $n \times n$ matrix, the determinant of M is given by

$$\det(M) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n M_{i, \sigma(i)}$$

where S_n denotes the set of permutations of $\{1, 2, \dots, n\}$; a transposition is a permutation that interchanges exactly two distinct elements of $\{1, 2, \dots, n\}$ while leaving all other elements fixed, any $\sigma \in S_n$ is the composition of transpositions. The sign function on S_n is given by $\text{sgn}(\sigma) = \pm 1$, with value 1 exactly when σ is the composition of an even number of transpositions.

4. Let A be a nilpotent matrix (A^k is the zero matrix, for some positive integer k). Prove that the trace of A is zero.

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5. Let A be a nonsingular real $n \times n$ matrix. Prove that there exists a unique orthogonal matrix Q and a unique positive definite symmetric matrix B such that $A = QB$. (Recall that B is positive definite if it is self-adjoint and $\langle Bx, x \rangle > 0$ for all nonzero $x \in \mathbb{R}^n$.)
6. Let $T : V \rightarrow V$ be a linear operator on a complex finite dimensional vector space. Let V^* denote the dual space of V , and T^t the transpose linear operator for T . Using component-wise operations, form the external direct sum vector space $V \oplus V^*$; let $\tilde{T} : V \oplus V^* \rightarrow V \oplus V^*$ be given by $\tilde{T}(v, \ell) = (Tv, T^t\ell)$. Determine, with proof, the minimal polynomial of \tilde{T} .