OSU Department of Mathematics
Qualifying Examination
Summer 2021

Real Analysis

Instructions:

- Do any **four of the six problems**.
- Use separate sheets of paper for each problem. Clearly indicate the problem and page number (if several pages are used for a solution) on the top of the page.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete this examination.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination:
  1. Use a separate sheet of paper to clearly indicate your identification number and the four problems which you wish to be graded.
  2. Arrange your solutions according to the problem order with the problem selection selection page on top and any scratch-work on the bottom.
  3. Submit the exam:
     - **For the in-person exam:** place your solutions together with the selection sheet and scratch paper, in the order arranged as above, into the envelope in which you received the exam and submit it to the proctor.
     - **For the on-line exam:**
       * scan your exam in the order arranged as above, starting with the selection page and ending with the scratch-work, as a single pdf file (using e.g. CamScan phone app);
       * check that your scan is legible and contains all the necessary pages to be graded;
       * email the file directly to Nichole Sullivan (Nikki.Sullivan@oregonstate.edu);
       * wait online until it is confirmed that your submission was received.

Exam continues on next page ...

Common Notation:

- $C^k(I)$ is the set of all functions on an interval I that have continuous derivatives up to and including order k.
- $\|a_n\|_p$ is the p-norm of a sequence (a_n): $\|a_n\|_p = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p}$, for $1 \leq p < \infty$, and $\|a_n\|_\infty = \sup_{n\in\mathbb{N}} |a_n|$, $p = \infty$.
- l_p – the normed space of all p-summable sequences: $l_p = \{(a_n) : \|a_n\|_p < \infty\}$, $1 \leq p \leq \infty$.

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Problems:

1. Let $f : [0, +\infty) \to \mathbb{R}$ be a bounded and differentiable function. Let $f_\infty := \liminf_{t\to\infty} f(t)$ and $f^\infty := \limsup_{t\to\infty} f(t)$. Assume that $f_\infty \neq f^\infty$.
(Recall that $\liminf_{t\to\infty} f(t) = \sup_{\tau\geq 0} \inf\{f(t)|t \geq \tau\}$ and $\limsup_{t\to\infty} f(t) = \inf_{\tau\geq 0} \sup\{f(t)|t \geq \tau\}$.)

   (a) (3pt) Show that for each $\alpha \in (f_\infty, f^\infty)$, and for each $T \geq 0$, there exist s_1, u and s_2 with $T < s_1 < u < s_2$, such that $f(s_1) < \alpha < f(u)$ and $f(s_2) < \alpha$.

   (b) (3pt) Show that for each $\alpha \in (f_\infty, f^\infty)$, and for each $T \geq 0$, there exists some $t > T$ such that $\alpha < f(t)$ and $f'(t) = 0$.

   (c) (4pt) Show that there exists a sequence $t_n \to \infty$ such that $f(t_n) \to f^\infty$, and such that $f'(t_n) = 0$ for all n.

2. (10pt) Set
$$X = \{f \in C([0, 1]) : 0 \leq f(x) \leq 1 \text{ for all } x \in [0, 1]\}.$$
Show that there exists a unique function $u \in X$ satisfying the integral equation
$$u(x) - \frac{1}{4}\int_0^1 (x + y)u^2(y)\, dy = \frac{1}{2}\ .$$

3. (10pt) Let K be a compact subset of a metric space (X, d), and let $\mathcal{U}$ be an open cover for K. Show that there exists $\epsilon > 0$ such that for each $x \in K$, the ball
$$B_\epsilon(x) = \{y \in X|\ d(x, y) < \epsilon\}$$
is contained in some $A \in \mathcal{U}$.

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4. (10pt) Let (X,d) be a compact metric space, and let $f_n : X \to \mathbb{R}$ be an increasing sequence (i.e. $f_n(x) \leq f_{n+1}(x)$ for all $n \in \mathbb{N}$ and $x \in X$) of continuous functions which converge pointwise on X to a continuous function $f : X \to \mathbb{R}$. Show that $f_n \to f$ uniformly on X.

5. Let $p \in [1, \infty)$. Define

$$C_p = \left\{ f = (\beta_k) \in l_p | \; \exists M > 0 \text{ such that } \forall \; k \in \mathbb{N} \; |\beta_k|^p \leq \frac{M}{k^2} \right\},$$

and

$$C_{p,1} = \left\{ f = (\beta_k) \in l_p | \; \forall \; k \in \mathbb{N} \; |\beta_k|^p \leq \frac{1}{k^2} \right\}.$$

(a) (5pt) Prove that C_p is not closed in l_p.

(b) (5pt) Prove that $C_{p,1}$ is compact in l_p.

6. Let $f : [0, \infty) \to \mathbb{R}$ be differentiable function satisfying

$$f'(x) = -2f(x) + f(2x), \quad x \in [0, \infty) .$$

Moreover, assume that for any $n = \{0, 1, 2, \ldots\}$

$$\mu_n = \int_0^\infty x^n f(x) \, dx$$

is well-defined (the improper integral is convergent) and finite.

(a) (6pt) Show that for any $n \in \mathbb{N}$ $\mu_n = \frac{n!}{2^n} \prod_{j=3}^{n+2} \frac{2^j}{2^j - 1} \mu_0$.

(b) (4pt) Prove that $\mu_n (2^n / n!)$ is convergent as $n \to \infty$.