Department of Mathematics OSU
Qualifying Examination
Spring 2019

Real Analysis

- Do any of the four of the six problems. Indicate on the sheet with your identification number the four which you wish graded.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have three hours to complete this examination.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination, place examination blue book(s) and selection sheet into the envelope in which the exam came. You will hand in all materials. If you use extra examination books, be sure to place your code number on them.

1. Consider $f(x) = 4^{-x}$, $x \in (-\infty, \infty)$. Define $f_1 = f$ and, recursively, $f_{n+1} = f \circ f_n$, thus

$$f_n(x) = [\underbrace{f \circ f \circ \cdots \circ f}_{n\,\text{times}}](x).$$

Prove that the sequence $\{f_n(x)\}_{n \in \mathbb{N}}$ converges uniformly on $(-\infty, \infty)$ and find its limit.

2. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map between the Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$. Let $\mathcal{C}$ be a closed subset of $\mathbb{R}^n$ which is also a *cone* (so that if $x \in \mathcal{C}$, then $\alpha x \in \mathcal{C}$ for all real $\alpha \geq 0$). Prove that if

$$\mathrm{Ker}(T) \cap \mathcal{C} = \{0\},$$

then $T(\mathcal{C})$ is a closed cone in $\mathbb{R}^m$. Here, $\mathrm{Ker}(T)$ is the null-space of T.

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3. Let $\mathcal{C}_0 = \{f : \mathbb{R} \to \mathbb{R} \mid f$ is continuous and $\lim_{x\to\pm\infty} f(x) = 0\}$. You may assume that this space is complete when it is equipped with the usual norm

$$|f|_{\mathcal{C}_0} = \sup\{|f(x)| : x \in \mathbb{R}\}.$$

Let $\mathcal{F} = \{f_n \mid n \in \mathbb{N}\}$, where $f_n : \mathbb{R} \to \mathbb{R}$ is defined by

$$f_n(x) = \begin{cases} 1 & \text{if } |x| < n, \\ \dfrac{n}{|x|} & \text{if } |x| \geq n. \end{cases}$$

(a) Show that $\mathcal{F}$ is a bounded equicontinuous family of functions in $\mathcal{C}_0$.

(b) Show that $\mathcal{F}$ contains no uniformly convergent (sub)sequences. Explain why this is not a contradiction to the Arzelà-Ascoli Theorem.

(c) A set $\mathcal{S} \subset \mathcal{C}_0$ is said to be *tight* if for every $\epsilon > 0$ there exists $R > 0$ such that

$$\forall f \in \mathcal{S}, \quad |x| > R \quad \longrightarrow \quad |f(x)| < \epsilon.$$

Show that if $\mathcal{S}$ is a bounded, equicontinuous and tight subset of $\mathcal{C}_0$, then its closure is compact.

4. Fix $p \in [1, \infty)$, and let $\{\mathbf{a}_n\}_{n\in\mathbb{N}}$ be a sequence of sequences in ℓ_p; thus, for each $n \in \mathbb{N}$, $\mathbf{a}_n = \{a_n^{(k)}\}_{k\in\mathbb{N}}$ is a sequence in ℓ_p. Assume that $\mathbf{a}_n$ converges componentwise to a sequence $\mathbf{a} = \{a^{(k)}\}_{k\in\mathbb{N}}$; thus, for any $k \in \mathbb{N}$, $\lim_{n\to\infty} a_n^{(k)} = a^{(k)}$.

(a) Show that (*i.*) $\mathbf{a}$ is not necessarily in ℓ_p; and (*ii.*) even if we assume $\mathbf{a}$ is in ℓ^p, then it does not follow that $\mathbf{a}_n$ converges to $\mathbf{a}$ in ℓ_p.

(b) Suppose that, in addition to the above, there exists a sequence $\mathbf{b} = \{b^{(k)}\}_{k\in\mathbb{N}}$ in ℓ_p such that for all $n, k \in \mathbb{N}$, $|a_n^{(k)}| \leq b^{(k)}$. Prove that in this case $\mathbf{a} \in \ell_p$ and the sequence $\mathbf{a}_n$ converges to $\mathbf{a}$ in ℓ_p

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5. Assume $f : \mathbb{R} \to \mathbb{R}$ is a continuous non-negative function such that $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = M < \infty$. Show that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f(x+1/n) - f(x)|\,\mathrm{d}x = 0.$$

6. Let (X, d) be a metric space.

   (a) Suppose that every function $f : (X, d) \to (X, d)$ is continuous. Prove that any subset of X is open in (X, d).

   (b) Suppose that the only continuous functions $f : (X, d) \to (\mathbb{R}, |\cdot|)$ are constant functions. Prove that X contains only one point.

   (Note: you must provide direct proofs, e.g. if you decide to use advanced results from topology outside the basic syllabus, you must also attach the proofs of such results!)