Department of Mathematics
Special Qualifying Examination
Spring 2007

Part I: Real Analysis

- Do any four of the problems in Part I.
- Your solutions should include all essential mathematical details; please write them up as clearly as possible.
- State explicitly including all hypotheses any standard theorems that are needed to justify your reasoning.
- You have three hours to complete Part I of the exam.
- In problems with multiple parts, the individual parts may be weighted differently in grading.

1. Let H be a (not necessarily separable) Hilbert space, with inner product $\langle\ ,\ \rangle$. Let $\{e_\alpha\ |\alpha \in \Lambda\}$ be an orthonormal collection of elements in H; that is

$$\langle e_\alpha, e_\beta \rangle = \begin{cases} 0 & \text{if } \alpha \neq \beta; \\ 1 & \text{if } \alpha = \beta. \end{cases}$$

Show that for every given $x \in H$, $\langle x, e_\alpha \rangle \neq 0$ for at most countably many α.

2. Let f be a Lebesgue integrable function on R^n i.e. $\int_{R^n} |f(x)| d\mu(x) < \infty$ where $\mu(x)$ is the Lebesgue measure on R^n. For every $x \in R^n$, define

$$G(x) = \int_{\{y \in R^n|\ |y-x|<1\}} f(y) d\mu(y).$$

Show that G is a continuous function on R^n.

3. Suppose f is a nonnegative function defined on a Lebesgue measurable subset E of R^n. Show that the set

$$\{(x,t) : 0 \leq t \leq f(x),\ \ x \in E\}$$

is Lebesgue measurable on R^{n+1} if and only if f is a Lebesgue measurable function on E.

4. A sequence $\{f_k\}$ in $L^2(R^n)$ is said to converge weakly to a function f in $L^2(R^n)$ if

$$\lim_{k\to\infty} \int f_k(x) g(x) d\mu(x) = \int f(x) g(x) d\mu(x)$$

for all $g \in L^2(R^n)$ where $d\mu(x)$ is the Lebesgue measure on R^n.

   (a) Show that if a sequence $\{f_k\}$ is convergent in $L^2(R^n)$, then the sequence is weakly convergent.

   (b) Give a counterexample to show the converse of (a) is not true.

Continued on next page

5. Suppose f is a measurable function on $[0,1]$ and $0 < f(x) < \infty$ for all $x \in [0,1]$. What is the least upper bound of the set of all A such that for all f as above

$$A \leq \left(\int_0^1 |f(x)| dx \right) \left(\int_0^1 \frac{1}{|f(x)|} dx \right) ?$$

Hint: Start from the simple function. You may need the following inequality. Let a_k, $k = 1, 2, \cdots, m$ be a finite sequence of positive real numbers and let ϵ_k, $k = 1, 2, \cdots, m$ be a finite sequence of real numbers satisfying $0 \leq \epsilon_k \leq 1$, $k = 1, 2, \cdots, m$ and $\epsilon_1 + \epsilon_2 + \cdots + \epsilon_m = 1$. Then

$$a_1^{\epsilon_1} a_2^{\epsilon_2} \cdots a_m^{\epsilon_m} \leq a_1 \epsilon_1 + a_2 \epsilon_2 + \cdots + a_m \epsilon_m.$$

6. Let $\mathcal{C}([0,1])$ denote the collection of all continuous functions on $[0,1]$. A real-valued function f on $[0,1]$ is said to be Hölder continuous of order α if there is a constant C such that $|f(x) - f(y)| \leq C|x - y|^{\alpha}$. Define

$$\|f\|_{\alpha} = \max_{x \in [0,1]} |f(x)| + \sup_{x,y \in [0,1]} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}}.$$

Show that, for $0 < \alpha \leq 1$, the set

$$\{ f \in \mathcal{C}([0,1]) : \|f\|_{\alpha} \leq 1 \}$$

is a compact subset of $\mathcal{C}([0,1])$.

Department of Mathematics
Special Qualifying Examination
Spring 2007

Part II: Complex Analysis and Linear Algebra

- Do any two problems in Part CA and any two problems in Part LA.
- Your solutions should include all essential mathematical details; please write them up as clearly as possible.
- State explicitly including all hypotheses any standard theorems that are needed to justify your reasoning.
- You have three hours to complete Part II of the exam.
- In problems with multiple parts, the individual parts may be weighted differently in grading.

Part: Complex Analysis

1. Let $p(z)$ denote the complex polynomial $\sum_{k=0}^{n}(k+1)z^k$, of degree n, $n \geq 1$. Let α be a positive real number. Evaluate the integral

$$\int_{|z|=\alpha} z^{n-1}|p(z)|^2 dz.$$

2. Let S_r denote the circle centered at 0 with radius r in the complex plane, i.e. $S_r = \{z \in \mathbb{C}|\ |z| = r\}$. Show that there does not exist an analytic function which maps the annulus $\{z|\ 8 \leq |z| \leq 27\}$ **onto** the annulus $\{z|\ 2 \leq |z| \leq 3\}$ with $S_8 \to S_2$ (mapping inner boundary to inner boundary) and $S_{27} \to S_3$ (outer boundary to outer boundary).

3. Suppose that f is analytic in the open unit disc D and $f(\frac{1}{n^2}) = f''(\frac{1}{n^2})$ for all $n \in \mathbb{N}$. Show that f is the restriction to the disc of an entire function.

Continued on next page

Part: Linear Algebra

1. Let A be an $n \times n$ matrix with real entries. Suppose A is nilpotent, i.e. A^m is the zero matrix for some $m \geq 1$, and $B = c_0 I_n + c_1 A + \cdots + c_{m-1} A^{m-1}$, where $c_0, \cdots, c_{m-1} \in \mathbf{R}$ and I_n is the $n \times n$ identity matrix. Show that $\det(B) = 0$ if and only if $c_0 = 0$.

2. Let A be a normal matrix. Show that there exists a polynomial p such that $A^* = p(A)$ where A^* is the conjugate transpose (i.e. Hermitian adjoint) of A.

3. Suppose A is a complex $n \times n$ matrix which satisfies the matrix equation $A^k = I_n$ for some positive integer k. Prove that A is diagonalizable.