OSU Department of Mathematics
Qualifying Examination
Fall 2022

Real Analysis

Instructions:

- Do **any three** of the four problems.
- Use separate sheets of paper for each problem. Clearly indicate the problem and page number (if several pages are used for a solution) on the top of the page.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have **four** hours to complete this examination.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination:
  1. Use the problem selection sheet to indicate your identification number and the three problems which you wish to be graded.
  2. Arrange your solutions according to the problem order with the problem selection sheet on top and any scratch-work on the bottom.
  3. Submit the exam: place your solutions together with the selection sheet and scratch paper, in the order arranged as above, into the envelope in which you received the exam and submit it to the proctor.

Exam continues on next page ...

Common notation:

- In the context of metric spaces, the notation (M,d) denotes a metric space M with metric d.

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Problems:

1. Let X be a normed vector space and $x_n \in X$, $n = 1, 2, \ldots$.

   a. (3 pts) Assume that X is complete. Show that X has the following property:

   $$\text{A series } \sum_{n=1}^{\infty} x_n \text{ converges in } X \text{ whenever } \sum_{n=1}^{\infty} \|x_n\| < \infty. \tag{1}$$

   As usual, $\|x_n\|$ denotes the norm of x_n.

   b. (7 pts) Assume that X has the property (1). Show that X is complete.

2. Consider a Banach space V, a contraction map $T : V \to V$, and the equation

   $$v = T(v) + y. \tag{2}$$

   The operator T is not necessarily linear.

   a. (3 pts) Show that for any $y \in V$, the solution to (2) exists and is unique.

   b. (3 pts) Based on (a), call $u(y)$ this solution to (2). Show that $u(y)$ is a continuous function of y.

   c. (4 pts) Show that if $T : K \to K \subset V$, with $T(0) = 0$, and $K = \{v \in V : \|v\| \leq r\}$, with a fixed $r > 0$, then the unique solution $u(y)$ of (2) lies in K, assuming $\|y\|$ is sufficiently small.

3. Let V be a Banach space, and $v_0, v_1 \in V$, $\lambda \in \mathbb{R}$. Define a sequence recursively by

   $$v_{n+1} = \lambda v_n + (1 - \lambda) v_{n-1}.$$

   a. (7 pts) Show that if $0 < \lambda < 2$, the sequence converges.

   b. (3 pts) Find conditions on v_0, v_1, λ that are sufficient and necessary for the sequence v_n to converge.

4. **a.** (6 pts) Prove that a compact metric space (M, d) is separable.

   b. (4 pts) Consider the normed vector space $C(\mathbb{R})$ of all bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$ equipped with the supremum norm $\|f\| = \sup_{x \in \mathbb{R}} |f(x)|$. Prove that $C(\mathbb{R})$ is not separable.

   Hint for part b.: Assume that $C(\mathbb{R})$ is separable. Show that then there exists a countable subset $\{f_n, n = 1, 2, \ldots\}$ of $C(\mathbb{R})$ such that for each $g \in C(\mathbb{R})$ there exists $n_0 \in \mathbb{N}$ such that $\|g - f_{n_0}\| < \frac{1}{2}$. Then derive a contradiction by constructing a function $g \in C(\mathbb{R})$ that violates this property.