OSU Department of Mathematics
Qualifying Examination
Spring 2023

Real Analysis

Instructions:

- Do **any three** of the four problems.
- Use separate sheets of paper for each problem. Clearly indicate the problem and page number (if several pages are used for a solution) on the top of the page.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have **four** hours to complete this examination.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination:
  1. Use the problem selection sheet to indicate your identification number and the three problems which you wish to be graded.
  2. Arrange your solutions according to the problem order with the problem selection sheet on top and any scratch-work on the bottom.
  3. Submit the exam: place your solutions together with the selection sheet and scratch paper, in the order arranged as above, into the envelope in which you received the exam and submit it to the proctor.

Exam continues on next page ...

Common notations:

- $C(X)$ denotes the space of all real-valued continuous functions on metric space (X, d).

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Problems:

1. Let $(g_n)_{n=1}^{\infty}$ be a sequence of continuous real-valued functions on $[0, \infty)$, and assume that there exists a real number $M > 0$ such that $|g_n(x)| \leq M$ for all $x \geq 0$ and all $n \geq 1$. Then let $F_n(x) = \int_0^x g_n(t)\, dt$ for all $x \geq 0$ and all $n \geq 1$.

   a. (4 pts) Prove that for every $B > 0$, there exists a subsequence $(F_{n_k})_{k=1}^{\infty}$ that converges uniformly on $[0, B]$.

   b. (4 pts) The subsequence in part (a) can depend on the choice of B. Now prove that there exists a subsequence $(F_{n_j})_{j=1}^{\infty}$ that is independent of B. In other words, prove that there exists a subsequence $(F_{n_j})_{j=1}^{\infty}$ such that, for every $B > 0$, the subsequence converges uniformly on $[0, B]$.

   c. (2 pts) Must the subsequence found in part (b) converge uniformly on $[0, \infty)$? Either prove that this is the case, or provide a counterexample with an explanation.

2. Let V be a vector space with norm $\|\cdot\|$, and let $T : V \to \mathbb{R}$ be a linear map. (Here, "linear" means $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all $x, y \in V$ and all $\alpha, \beta \in \mathbb{R}$.) For any $x \in V$ and any $r > 0$, let $B_r(x)$ denote the open ball $\{y \in V : \|y - x\| < r\}$.

   a. (1 pt) Prove that $T(0) = 0$. Here, the symbol "0" on the left side is the identity element for addition in V, and symbol "0" on the right side is $0 \in \mathbb{R}$.

   b. (3 pts) Assume that there exists $r > 0$ and $M > 0$ such that $|T(x)| < M$ for all $x \in B_r(0)$; in other words, $\|x\| < r$ implies $|T(x)| < M$. Prove that T is continuous at $0 \in V$.

   c. (3 pts) Under the same assumption as in part (b), prove that T is continuous at every point in V.

   d. (3 pts) State an example of a vector space V, a norm $\|\cdot\|$ on V, and a linear map $T : V \to \mathbb{R}$ such that T is discontinuous at one or more points in V. Explain why your example has the required properties. When constructing your example, make sure that $T(f)$ is a well-defined real number, for every $f \in V$.

3. Let $g : [0, 1] \to \mathbb{R}$ be continuous. Show that there is a unique continuous function $f : [0, 1] \to \mathbb{R}$ such that

$$f(x) + \int_0^x f(t) \sin(\pi t/4) dt = g(x), \quad 0 \leq x \leq 1.$$

   Be sure to verify all hypotheses of any theorems you are using.

4. **a.** (3 pts) Let (X, d) and (Y, ρ) be metric spaces, let $f : X \to Y$ be a function, and let $(f_n)_{n=1}^{\infty}$ be a sequence of functions mapping X into Y. Prove that if $(f_n)_{n=1}^{\infty}$ converges uniformly to f on X, and if each f_n is continuous at $x \in X$, then f is also continuous at x.

   b. (7 pts) Let (X, d) be a compact metric space, and suppose that the sequence $(f_n)_{n=1}^{\infty}$ in $C(X)$ increases pointwise to a *continuous* function $f \in C(X)$; that is $f_n(x) \leq f_{n+1}(x)$ for each n and x, and $f_n(x) \to f(x)$ for each x. Prove that the convergence is uniform.