ANALOGUES OF ALDER-TYPE PARTITION INEQUALITIES FOR FIXED PERIMETER PARTITIONS

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Abstract. In a 2016 paper, Straub proved an analogue to Euler's celebrated partition identity for partitions with a fixed perimeter. Later, Fu and Tang provided both a refinement and generalization of Straub's analogue to d-distinct partitions. They also prove a related result to the first Rogers-Ramanujan identity by defining two new functions, $h_d(n)$ and $f_d(n)$ for a fixed perimeter n, that resemble the preexisitng q_d and Q_d functions. Motivated by generalizations of Alder's ex-conjecture, we further generalize the work done by Fu and Tang by introducing a new parameter a, similar to the work of Kang and Park. We observe the prevalence of binomial coefficients in our study of fixed perimeter partitions and use this to develop a more direct analogue to Q_d. Using combinatorial techniques, we find Alder-type partition inequalities in a fixed perimeter setting, specifically a reverse Alder-type inequality.

1. Introduction and Statement of Results

Given a positive integer n, a *partition* π of size n is a nonincreasing sequence of positive integers called *parts* that sum to n. For any positive integer n, we define the function $p(n)$ to count the total number of partitions of size n. We will use $p(n|\text{condition})$ to denote the number of partitions of size n satisfying the specified condition.

In 1748, Euler proved a celebrated partition theorem, which says that for any positive integer n, the number of partitions of size n into distinct parts is equal to the number of partitions of size n into odd parts, namely,

$$p(n|\text{distinct parts}) = p(n|\text{odd parts}).$$

Furthermore, we have two important identities from Rogers and Ramanujan. The first identity says that the number of partitions of n into parts that are 2-distinct is equal to the number of partitions of n into parts congruent to ± 1 modulo 5:

$$p(n|\text{2-distinct parts}) = p(n|\text{parts} \equiv \pm 1 \pmod 5).$$

The second Rogers-Ramanujan identity says that the number of partitions of n into parts that are 2-distinct and greater than or equal to 2 is equal to the number of partitions of n into parts congruent to ± 2 modulo 5:

$$p(n|\text{2-distinct parts} \geq 2) = p(n|\text{parts} \equiv \pm 2 \pmod 5).$$

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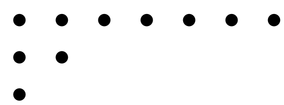
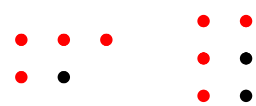
Additionally, a theorem from Schur says that the number of partitions of n into parts that are 3-distinct is greater than or equal to the number of partitions of n into parts congruent to ± 1 modulo 6:

$$p(n|\text{3-distinct parts}) \geq p(n|\text{parts} \equiv \pm 1 \pmod{6}).$$

A useful tool for visualizing a partition π is through the use of a *Ferrers diagram*, in which each part π_i is represented by a row of π_i dots. From top to bottom, the rows are arranged in nonincreasing order and left-justified. For example, the following diagram represents the partition $\pi = (7, 2, 1)$.



For any partition π, we let $\alpha(\pi)$ denote the largest part and $\lambda(\pi)$ to denote the number of parts. We can now define the *perimeter* of a partition to be the largest hook length of π, denoted $\Gamma(\pi)$, which is also given by the number of dots traversing the top row and the left column of its Ferrers diagram. In other words, $\Gamma(\pi) = \alpha(\pi) + \lambda(\pi) - 1$. As an example, we show below the Ferrers diagrams of two partitions with perimeter 4.



As an analogue to the traditional partition function $p(n)$, we define $r(n)$ to output the total number of partitions with fixed perimeter n. In 2016, Straub proved the following analogue to Euler's Theorem using the idea of fixed perimeter.

Theorem 1.1 (Straub [10], 2016)**.** *The number of partitions with perimeter n into distinct parts is equal to the number of partitions of with perimeter n into odd parts.*

Based on Euler's identity along with the first Rogers-Ramanujan identity, Alder considered the following two functions,

$$q_d(n) = p(n|\text{d-distinct parts}),$$

$$Q_d(n) = p(n|\text{parts} \equiv \pm 1 \pmod{d+3}).$$

Note that we can now restate Euler's identity as $q_1(n) = Q_1(n)$ and the first Rogers-Ramanujan identity as $q_2(n) = Q_2(n)$. Additionally, Schur's theorem can be rewritten as $q_3(n) \geq Q_3(n)$. This pattern led to a famous conjecture by Alder, which says that $q_d(n) \geq Q_d(n)$ for all positive integers d and n. This conjecture was proved over the span of several decades. First, Andrews [2] proved the inequality for $d = 2^r - 1, r \geq 4$ in 1971. Then, in 2008, Yee proved the conjecture for $d \geq 32$ and $d = 7$. Finally, the remaining cases were proved by Alfes, Jameson, and Lemke Oliver [1] in 2011.

Later, Kang and Park [9] generalized $q_d(n)$ and $Q_d(n)$ as follows to account for the second Rogers-Ramanujan Identity,

$$q_d^{(2)}(n) = p(n|\text{parts are d-distinct and} \geq 2),$$

$$Q_d^{(2)}(n) = p(n|\text{parts are} \equiv \pm 2 \pmod{d+3}).$$

This allows us to restate the second Rogers Ramanujan identity as $q_2^{(2)}(n) = Q_2^{(2)}(n)$. However, it is *not* true that $q_d^{(2)}(n) \geq Q_d^{(2)}(n)$ for all d and n, so Kang and Park introduced a modified version of $Q_d^{(2)}(n)$,

$$Q_d^{(2,-)}(n) = p(n|\text{parts are} \equiv \pm 2 \pmod{d+3} \text{ excluding the part } d+1).$$

With this new definition in mind, Kang and Park [9] conjectured that $q_d^{(2)}(n) \geq Q_d^{(2,-)}(n)$ for all positive integers d and n, which they were able to prove for $d = 2^r - 2, r = 1, 2,$ or $r \geq 4$.

After this, Duncan, Khunger, Swisher and Tamura [4] generalized Yee's proof to prove Kang and Park's conjecture for $d \geq 62$. Finally, Sturman and Swisher used asymptotics to prove all remaining cases except $d = 3, 4, 5,$ and 7. Duncan, Khunger, Swisher, and Tamura also generalized $q_d^{(2)}(n)$ and $Q_d^{(2)}(n)$ to

$$q_d^{(a)}(n) = p(n|\text{parts are d-distinct and} \geq a),$$

$$Q_d^{(a)}(n) = p(n|\text{parts} \equiv \pm a \pmod{d+3}).$$

Further work was done by Inagaki and Tamura [7] generalizing techniques to higher values of a. Armstrong, Ducasse, Meyer, and Swisher [3] also investigated general shifts of both a and d. Kang and Kim [8] compared $q_d^{(a)}(n)$ and $Q_d^{(a)}(n)$ along with the variations allowing the parameters of each term to vary independently and discovered results about general inequalities based on $a, d,$ and n.

In the style of Alder, Fu and Tang generalized Straub's analogue to Euler's identity by introducing the following functions $h_d(n)$ and $f_d(n)$,

$$h_d(n) = r(n|\text{parts are d-distinct}),$$

$$f_d(n) = r(n|\text{parts are} \equiv 1 \,(\text{mod } d+1)).$$

They were able to find generating functions for both $h_d(n)$ and $f_d(n)$ and prove the following theorem.

Theorem 1.2 (Fu and Tang [6] 2018)**.** *For all $n, d \in \mathbb{N}$,*

$$h_d(n) = f_d(n).$$

Fu and Tang also proved a refinement of their result for $d = 1$.

Theorem 1.3 (Fu and Tang [6] 2018)**.** *The number of partitions counted by $h_1(n)$ with exactly k parts is equal to the number of partitions counted by $f_1(n)$ with largest part of size $2k - 1$ and both are enumerated by $\binom{n-k}{k-1}$.*

1.1. **Our Results.** As an attempt to find analogues to Kang and Park's conjecture in a fixed perimeter setting, we introduce the parameter a to the functions $h_d(n)$ and $f_d(n)$. We start by extending the definition for $h_d(n)$ to obtain the following.

Definition 1.4. *For any positive integer n and positive integers a and d, we define*

$$h_d^{(a)}(n) = r(n|\textit{parts are d-distinct and } \geq a),$$

and consider the refinement

$$h_d^{(a)}(\alpha, \lambda, n) = r(n|\textit{parts are d-distinct, } \geq a \textit{ with largest piece } \alpha \textit{ and } \lambda \textit{ parts}).$$

We define $\mathcal{H}_d^{(a)}(n)$ to be the set of all partitions with perimeter n and parts that are d-distinct and greater than or equal to a.

Similarly, we extend the definition of $f_d(n)$ to obtain the following.

Definition 1.5. *For any positive integer n and positive integers a and d, we define*

$$f_d^{(a)}(n) = r(n|\textit{parts are} \equiv a \pmod{d+1}),$$

and consider the refinement

$$f_d^{(a)}(\alpha, \lambda, n) = r(n|\textit{parts are} \equiv a \pmod{d+1} \textit{with largest piece } \alpha \textit{ and } \lambda \textit{ parts}).$$

We define $\mathcal{F}_d^{(a)}(n)$ to be the set of all partitions with perimeter n and parts congruent to a $\pmod{d+1}$.

These functions generalize $h_d(n)$ and $f_d(n)$ in the exact same way that $q_d^{(a)}(n)$ and $Q_d^{(a)}(n)$ generalize $q_d(n)$ and $Q_d(n)$. We also note that $f_d^{(a)}(\alpha, \lambda, n)$ is nonzero only if $\alpha \equiv a \pmod{d+1}$. Along with the extensions of the functions originally defined by Fu and Tang [6], we attempt to find a more direct analogue of $Q_d^{(a)}(n)$. This leads us to the definition of the following function.

Definition 1.6. *For any positive integer n and positive integers a and d, we define*

$$\ell_d^{(a)}(n) = r(n|\textit{parts are} \equiv \pm a \pmod{d+3}),$$

and consider the refinement

$$\ell_d^{(a)}(\alpha, \lambda, n) = r(n|\textit{parts are} \equiv \pm a \pmod{d+3} \textit{with largest piece } \alpha \textit{ and } \lambda \textit{ parts}).$$

We define $\mathcal{L}_d^{(a)}(n)$ to be the set of all partitions with perimeter n and parts congruent to $\pm a$ $\pmod{d+3}$.

We should note that while we write the refinements in Definitions 1.4 through 1.6 as functions of three parameters, since the perimeter is defined as $\alpha + \lambda - 1 = n$, specifying two parameters will allow us to solve for the third. We will switch between parameter variables for convenience, mainly rewriting $h_d^{(a)}(\alpha, \lambda, \alpha + \lambda - 1) = h_d^{(a)}(n - \lambda + 1, \lambda, n)$. We also define the following generating series.

Definition 1.7. *For positive integers $a, d,$ and n, let*

$$H_d^{(a)}(q) := \sum_{\pi \in \mathcal{H}_d^{(a)}} q^{\Gamma(\pi)} = \sum_{n=1}^{\infty} h_d^{(a)}(n) q^n,$$

$$F_d^{(a)}(q) := \sum_{\pi \in \mathcal{F}_d^{(a)}} q^{\Gamma(\pi)} = \sum_{n=1}^{\infty} f_d^{(a)}(n) q^n,$$

$$L_d^{(a)}(q) := \sum_{\pi \in \mathcal{L}_d^{(a)}} q^{\Gamma(\pi)} = \sum_{n=1}^{\infty} \ell_d^{(a)}(n) q^n,$$

$$H_d^{(a)}(x,y,q) = \sum_{\pi \in \mathcal{H}_d^{(a)}} x^{\alpha(\pi)} y^{\lambda(\pi)} q^{\Gamma(\pi)} = \sum_{\alpha=0}^{\infty} \sum_{\lambda=0}^{\infty} h_d^{(a)}(\alpha,\lambda,\alpha+\lambda-1) x^{\alpha} y^{\lambda} q^{\alpha+\lambda-1},$$

$$F_d^{(a)}(x,y,q) = \sum_{\pi \in \mathcal{F}_d^{(a)}} x^{\alpha(\pi)} y^{\lambda(\pi)} q^{\Gamma(\pi)} = \sum_{\alpha=0}^{\infty} \sum_{\lambda=0}^{\infty} f_d^{(a)}(\alpha,\lambda,\alpha+\lambda-1) x^{\alpha} y^{\lambda} q^{\alpha+\lambda-1},$$

$$L_d^{(a)}(x,y,q) = \sum_{\pi \in \mathcal{L}_d^{(a)}} x^{\alpha(\pi)} y^{\lambda(\pi)} q^{\Gamma(\pi)} = \sum_{\alpha=0}^{\infty} \sum_{\lambda=0}^{\infty} \ell_d^{(a)}(\alpha,\lambda,\alpha+\lambda-1) x^{\alpha} y^{\lambda} q^{\alpha+\lambda-1}.$$

We establish generating functions for $h_d^{(a)}(\alpha,\lambda,n)$ and $f_d^{(a)}(\alpha,\lambda,n)$ and use them to prove the following theorem.

Theorem 1.8. *For positive integers $1 \le d, 1 \le a \le d+1,$ and $n = \alpha+\lambda-1$,*

$$h_d^{(a)}(\alpha,\lambda,n) = \binom{n-a-(\lambda-1)(d)}{\lambda-1}, \text{ and } f_d^{(a)}(\alpha,\lambda,n) = \binom{\frac{n-a+d(\lambda-1)}{(d+1)}}{\lambda-1}.$$

We then use this fact along with generating functions to prove an analogue of Fu and Tang's Theorem.

Theorem 1.9. *For positive integers $1 \le d, 1 \le a \le d+1,$ and $a \le n$,*

$$h_d^{(a)}(n) = f_d^{(a)}(n).$$

Moreover, the number of partitions counted by $h_d^{(a)}(n)$ with k parts equals the number of partitions counted by $f_d^{(a)}(n)$ with largest part $a+(k-1)(d+1)$, namely,

$$h_d^{(a)}(n-\lambda+1,\lambda,n) = f_d^{(a)}(a+(d+1)(\lambda-1), n-(a+(d+1)(\lambda-1))+1, n).$$

We then find a sum formula for $\ell_d^{(a)}(n)$ and use it to prove the following reverse analogue to Alder's conjecture.

Theorem 1.10. *For positive integers $d, n,$ and $a < \frac{d+3}{2}$,*

$$h_d^{(a)}(n) \le \ell_d^{(a)}(n).$$

Lastly, we prove inequalities between $h_d^{(a)}(n)$ and $\ell_d^{(a)}(n)$.

Theorem 1.11. *For positive integers d, a, n,*

$$h_d^{(a)}(n) \ge h_d^{(a+1)}(n),$$
$$h_d^{(a)}(n) \ge h_{d+1}^{(a)}(n),$$
$$h_d^{(a)}(n) \le h_d^{(a)}(n+1),$$

and for positive integers $d, n,$ and $a < \frac{d+3}{2}$,

$$\ell_d^{(a)}(n) \le \ell_d^{(a)}(n+1),$$
$$\ell_d^{(a)}(n) \ge \ell_{d+1}^{(a)}(n).$$

We now outline the rest of the paper. In Section 2, we extend Fu and Tang's findings to include the parameter a and prove generating functions for $h_d^{(a)}(n)$ and $f_d^{(a)}(n)$. We then use our generating functions to prove Theorem 1.8, which we then use to prove Theorem 1.9. In Section 4, we will prove Theorem 1.10, and in Section 5 we prove Theorem 1.11. Then, in Section 6, we discuss a few remaining questions related to fixed perimeter. Finally, in section 7, we provide a brief note about our attempted approaches to proving the remaining cases of Kang and Park's conjecture.

2. Generating Functions

Fu and Tang [6] found generating functions for $h_d(n)$ and $f_d(n)$ given by

$$H_d^{(1)}(x,y,q) = \sum_{\alpha=0}^{\infty}\sum_{\lambda=0}^{\infty} h_d(\alpha,\lambda,\alpha+\lambda-1)x^\alpha y^\lambda q^{\alpha+\lambda-1} = \frac{xyq}{1-(xq+x^d yq^{d+1})}, \tag{1}$$

$$F_d^{(1)}(x,y,q) = \sum_{\alpha=0}^{\infty}\sum_{\lambda=0}^{\infty} f_d(\alpha,\lambda,\alpha+\lambda-1)x^\alpha y^\lambda q^{\alpha+\lambda-1} = \frac{xyq}{1-(yq+x^{d+1}q^{d+1})}. \tag{2}$$

They prove this result using a technique referred to as "word profiles" where they associate a choice of x as a movement to the right denoted by E and choice of y as a movement up denoted by N. This allows them to create a bijection between partitions of fixed perimeter and word profiles. As an example, Figure 1 shows how the partition $\pi = (2,2,1)$ may be interpreted using word profiles.

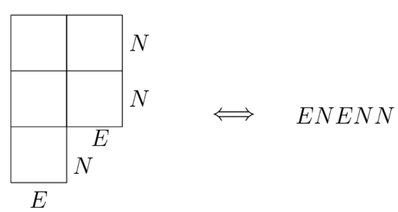


Figure 1. [6]
The profile of the $\pi = (2,2,1)$ labelled with E and N.

The generating functions have an xyq term in the numerator to represent the fact that all word profiles must begin with an E and end with a N. Then, since partitions counted by $h_d(n)$ must have d-distinct pieces, any N must be followed by at least d copies of E, giving us the term $x^d yq^{d+1}$. The xq term comes from the fact that movement to the right is unrestricted. Similarly, for $f_d(n)$, movement up is unrestricted, which gives us the term yq. However, movement to the right must be done in increments of $d+1$, which gives us the term $x^{d+1}q^{d+1}$. This proves that these generating functions give us the acceptable word profiles of the partitions counted by $h_d(n)$ and $f_d(n)$, thus proving that these are the correct generating functions.

Now, we wish to modify the generating functions given by (1) and (2) to include the parameter a. We start by proving the generating function for $h_d^{(a)}(n)$.

Proposition 2.1. *For positive integers* $1 \leq d, 1 \leq a$,

$$H_d^{(a)}(x,y,q) = \sum_{\alpha=0}^{\infty}\sum_{\lambda=0}^{\infty} h_d^{(a)}(\alpha,\lambda,\alpha+\lambda-1)x^\alpha y^\lambda q^{\alpha+\lambda-1} = \frac{x^a y q^a}{1-(xq+x^d y q^d)}.$$

Proof. Consider the generating function for $h_d^{(1)}(n)$ given by (1). We wish to modify this function such that we get word profiles corresponding to $h_d^{(a)}(n)$ instead of $h_d^{(1)}(n)$. Since we want our partitions to have parts of size at least a, we begin all word profiles with a copies of E instead of only 1. This gives us the term $x^a y q^a$ in the numerator instead of xyq. We see that all word profiles of this form are counted by $h_d^{(a)}(n)$ and all partitions counted will have their word profile appear. Therefore we conclude that $\frac{x^a y q^a}{1-(xq+x^d y q^d)}$ is the correct generating function for $h_d^{(a)}(n)$. □

Now, we apply the same techniques to prove the generating function for $f_d^{(a)}(n)$.

Proposition 2.2. *For positive integers* $1 \leq d$ *and* $1 \leq a \leq d+1$,

$$F_d^{(a)}(x,y,q) = \sum_{\alpha=0}^{\infty}\sum_{\lambda=0}^{\infty} f_d^{(a)}(\alpha,\lambda,\alpha+\lambda-1)x^\alpha y^\lambda q^{\alpha+\lambda-1} = \frac{x^a y q^a}{1-(yq+x^{d+1} q^{d+1})}.$$

Proof. This proof is the same as the proof of Proposition 2.1 as we follow Fu and Tang's result, but include $x^a y q^a$ instead of xyq in order for all parts to be congruent to a modulo $d+1$. □

Now, consider the generating functions given by Proposition 2.1 and Proposition 2.2 and set $x = y = 1$. Then we see that

$$H_d^{(a)}(1,1,q) = \sum_{n=1}^{\infty} h_d^{(a)}(n)q^n = \frac{q^a}{1-(q+q^d)},$$

and

$$F_d^{(a)}(1,1,q) = \sum_{n=1}^{\infty} f_d^{(a)}(n)q^n = \frac{q^a}{1-(q+q^d)}.$$

Thus since they have the same generating functions, we conclude that $H_d^{(a)}(1,1,q) = F_d^{(a)}(1,1,q)$, which implies all the coefficients are equal. Therefore for all positive $d, 1 \leq a \leq d+1$ and n,

$$h_d^{(a)}(n) = f_d^{(a)}(n). \tag{3}$$

3. Refinement Functions and Duality for $h_d^{(a)}(n)$ and $f_d^{(a)}(n)$

Equipped with our generating functions, we see that the expansions of terms lead us to refinement formulas in terms of binomial coefficients, which give us the tools to prove Theorem 1.8.

Proof of Theorem 1.8. First we establish a formula for $h_d^{(a)}(\alpha,\lambda,n)$. Given fixed λ and n, we wish to find the coefficient of $x^{n-\lambda+1}y^\lambda q^n$. We see that from the expansion of the generating series $x^a y q^a(1+(xq+x^d y q^{d+1})+(xq+x^d y q^{d+1})^2+(xq+x^d y q^{d+1})^3+\cdots)$ that the only terms which could contribute $\lambda-1$ copies of y and at most $n-a$ copies of q are those

$(xq+x^dyq^{d+1})^j$ such that $\lambda-1 \le j \le n-a$. Now consider the expansion of $(xq+x^dyq^{d+1})^j$ with $\lambda-1 \le j \le n-a$. If we choose $\lambda-1$ copies of x^dyq^{d+1}, we will end up with a term $x^{a+j-(\lambda-1)}y^\lambda q^{a+(\lambda-1)(d+1)+j}$. Thus if we want to find j such that $n = a+(\lambda-1)(d+1)+j$ then we see that $j = n-a-(\lambda-1)(d+1)$ thus the coefficient of $x^{n-\lambda}y^\lambda q^n$ is

$$\binom{n-a-(\lambda-1)(d)}{\lambda-1}.$$

Now we prove the coefficient for $f_d^{(a)}(\alpha,\lambda,n)$. Again, notice that λ and α are dependent on one another by $\alpha = n-\lambda+1$, and thus we only need to fix one of them, say λ. So given fixed λ and n, we wish to find the coefficient of $x^{n-\lambda+1}y^\lambda q^n$. Consider the expansion of the generating series for $F_d^{(a)}(x,y,q)$ given by

$$x^ayq^a(1+(yq+x^{d+1}q^{d+1})+(yq+x^{d+1}q^{d+1})^2+\cdots).$$

Notice that the term x^ayq^a contributes a copies of x, 1 copy of y, and a copies of q. Thus we wish to find some term $(yq+x^{d+1}q^{d+1})^j, \lambda-1 \le j \le n-a$ that contributes $\lambda-1$ copies of y and at most $n-a$ copies of q. If we choose $\lambda-1$ copies of yq, we obtain the term $x^{a+(j-(\lambda-1))(d+1)}y^\lambda q^{a+\lambda-1+(j-(\lambda-1))(d+1)}$. Thus we want to find j such that $n = a+\lambda-1+(j-(\lambda-1))(d+1) = a+j(d+1)-d(\lambda-1)$. Solving for j, we get $j = \dfrac{n-a+d(\lambda-1)}{d+1}$. Then $j \in \mathbb{N}$ when $n-a+d(\lambda-1) \equiv 0 \pmod{d+1}$ which implies $n-a-\lambda+1 \equiv 0 \pmod{d+1}$. Thus $n-\lambda+1 \equiv a \pmod{d+1}$ which occurs only when $\alpha \equiv a \pmod{d+1}$. But $\alpha \equiv a \pmod{d+1}$ for any partition counted by $f_d^{(a)}(n)$, so j is always a natural number. Therefore we have the coefficient of $x^{n-\lambda+1}y^\lambda q^n$ is

$$\binom{\frac{n-a+d(\lambda-1)}{(d+1)}}{\lambda-1}.$$

$\square$

Using these binomial coefficients, we can now give upper and lower bounds for λ and α for partitions counted by $h_d^{(a)}(n)$.

Proposition 3.1. *For any partition counted by $h_d^{(a)}(n)$, we have $1 \le \lambda \le \lfloor \frac{n-a}{d+1} \rfloor + 1$ and $n - \lfloor \frac{n-a}{d+1} \rfloor \le \alpha \le n$ for positive d,a.*

Proof. From Theorem 1.8, we know that for given λ, n, $h_d^{(a)}(n-\lambda+1,\lambda,n) = \binom{n-a-(\lambda-1)d}{\lambda-1}$. In order for this to produce any values, it must be true that $n-a-(\lambda-1)d \ge \lambda-1$. Solving for λ, we have $\lambda \le \lfloor \frac{n-a+d+1}{d+1} \rfloor = \lfloor \frac{n-a}{d+1} \rfloor + 1$. When $\lambda = \lfloor \frac{n-a}{d+1} \rfloor + 1$, we obtain a lower bound for α. Since $\alpha = n-\lambda+1$, we now have $\alpha \ge n - (\lfloor \frac{n-a}{d+1} \rfloor + 1) + 1$. $\square$

Similarly, we can give upper and lower bounds for λ and α for partitions counted by $f_d^{(a)}(n)$.

Proposition 3.2. *For any partition counted by $f_d^{(a)}(n)$, we have $1 \le \lambda \le n-a+1$ and $a \le \alpha \le n$, where $\alpha \equiv a \pmod{d+1}$ for positive $d, 1 \le a \le d+1$.*

Proof. For $a < d+1$, the largest part $\alpha(\pi)$ is bounded on the left by a, the smallest value of $a \pmod{d+1}$, and on the right by n, to account for when $n \equiv a \pmod{d+1}$. Then when $\alpha = a$ we have an upper bound for λ. Since $\lambda = n - \alpha + 1$, we get $\lambda \leq n - a + 1$. Similarly, when $\alpha = n$, we obtain a lower bound for λ, that is, 1. $\square$

Now, using our results from Propositions 3.1 and 3.2, we may compare the number of possible arm lengths for partitions in $\mathcal{H}_d^{(a)}(n)$ and partitions in $\mathcal{F}_d^{(a)}(n)$.

Proposition 3.3. *For positive integers d and $a \leq d+1$, the number of possible arm lengths for partitions counted by $h_d^{(a)}(n)$ is equal to the number of possible arm lengths for partitions counted by $f_d^{(a)}(n)$.*

Proof. We see that $h_d^{(a)}(\alpha, \lambda, n)$ takes positive values for all $1 \leq \lambda \leq \lfloor \frac{n-a}{d+1} \rfloor + 1$ and that $f_d^{(a)}(\alpha, \lambda, n)$ takes positive values for $\alpha = a + (d+1)k$ for all $0 \leq k \leq \lfloor \frac{n-a}{d+1} \rfloor$. Thus we see that the number of possible arm lengths for partitions counted by $h_d^{(a)}(n)$ is exactly equal to the number of possible arm lengths for partitions counted by $f_d^{(a)}(n)$. $\square$

We are now ready to prove Theorem 1.9.

Proof of Theorem 1.9. Using Theorem 1.8, we see that

$$h_d^{(a)}(n - \lambda + 1, \lambda, n) = \binom{n - a - (\lambda - 1)d}{\lambda - 1}$$

and

$$f_d^{(a)}\left(a + (d+1)(\lambda - 1), n - (a + (d+1)(\lambda - 1)) + 1, n\right) = \binom{\frac{n-a+d(n-(a+(d+1)(\lambda-1))}{d+1}}{n - (a + (d+1)(\lambda - 1))},$$

which simplifies to

$$\binom{\frac{n-a+d(n-(a+(d+1)(\lambda-1)))}{d+1}}{n - (a + (d+1)(\lambda - 1))} = \binom{n - a - (\lambda - 1)d}{n - a - (d+1)(\lambda - 1))}.$$

Now, using the binomial theorem, we have

$$\binom{n - a - (\lambda - 1)d}{n - a - (d+1)(\lambda - 1))} = \binom{n - a - (\lambda - 1)d}{\lambda - 1}.$$

Thus we get a correspondence between coefficients of $h_d^{(a)}$ and $f_d^{(a)}$ which accounts for the arm and leg length of partitions. This gives a refinement to the equality given in (3); by the duality of the coefficients and by Proposition 3.3, along with the fact that since each arm and leg length pair counted by $h_d^{(a)}(n)$ has a corresponding arm and leg length pair counted by $f_d^{(a)}(n)$ with the same number of partitions satisfying the two pairs, we see that $h_d^{(a)}(n) = f_d^{(a)}(n)$. $\square$

As an example, Figure 2 displays the fixed perimeter correspondence between $h_1^{(2)}(9)$ and $f_1^{(2)}(9)$. On the left, we have possible arm and leg lengths for partitions counted by $h_d^{(2)}(n)$,

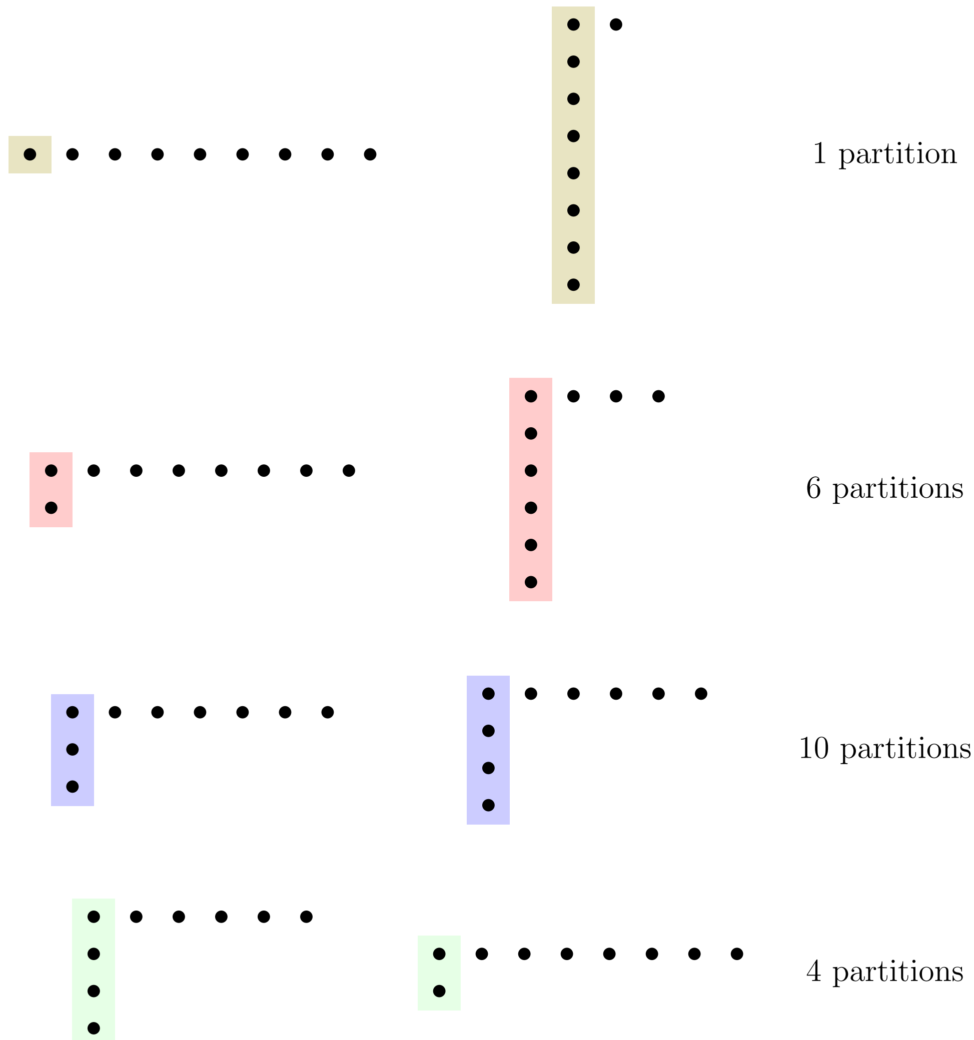


Figure 2.
Counting partitions for a fixed perimeter 9.

and on the right we have possible arm and leg lengths for partitions counted by $f_d^{(2)}(n)$. Leg lengths that are shaded the same color have the same number of associated partitions.

4. Reverse Alder Inequality

Now we wish to prove inequalities between $h_d^{(a)}(n)$ and $\ell_d^{(a)}(n)$. We start with a special case when $a = \frac{d+3}{2}$. When this occurs there is only one congruence class available for parts in partitions counted by $\ell_d^{(a)}(n)$.

Proposition 4.1. *For all positive integers n, d with d odd,*

$$\ell_d^{(\frac{d+3}{2})}(n) = f_{d+2}^{(\frac{d+3}{2})}(n).$$

Proof. We see that the two functions count the same partitions by their respective definitions, thus they are equal. ☐

We now use this fact, along with the equality between $h_d^{(a)}(n)$ and $f_d^{(a)}(n)$, to prove an Alder-type partition analogue for the special case when $a = \frac{d+3}{2}$.

Corollary 4.2. *For positive integers n, d with d odd, $\ell_d^{(\frac{d+3}{2})}(n) \leq h_d^{(\frac{d+3}{2})}(n)$.*

Proof. We know by Proposition 4.1 that $\ell_d^{(\frac{d+3}{2})}(n) = f_{d+2}^{(\frac{d+3}{2})}(n) = h_{d+2}^{(\frac{d+3}{2})}(n)$. We can also see that there is an injection from the partitions counted by $h_{d+1}^{(a)}(n)$ into the partitions counted by $h_d^{(a)}(n)$ as partitions that are $(d+1)$-distinct are also d-distinct. thus $\ell_d^{(\frac{d+3}{2})}(n) = h_{d+2}^{(\frac{d+3}{2})}(n) \leq h_d^{(\frac{d+3}{2})}(n)$. ☐

For all other cases, $\ell_d^{(a)}(\alpha, \lambda, n)$ counts partitions with parts that can be congruent to two different modulus classes. As a result, we prove two cases for the binomial coefficient formula for $\ell_d^{(a)}(\alpha, \lambda, n)$ based on arm lengths.

Proposition 4.3. *When $a < \frac{d+3}{2}$ with positive d,*

$$\ell_d^{(a)}(\alpha, \lambda, n) = \binom{2(\frac{\alpha-a}{d+3}) + 1 + (\lambda - 1) - 1}{\lambda - 1}$$

if $\alpha \equiv a \pmod{d+3}$, and

$$\ell_d^{(a)}(\alpha, \lambda, n) = \binom{2(\frac{\alpha+a}{d+3}) + (\lambda - 1) - 1}{\lambda - 1}$$

if $\alpha \equiv -a \pmod{d+3}$.

Proof. We see that we must pick $r = \lambda - 1$ elements from our acceptable list of elements $a, -a+(d+3), a+(d+3), -a+2(d+3), \ldots$ given by $k = 2(\frac{\alpha-a}{d+3})+1$ when $\alpha \equiv a \pmod{d+3}$ and $k = 2(\frac{\alpha+a}{d+3})$ when $\alpha \equiv -a \pmod{d+3}$. We can view this as putting r balls into k bins which by [5] we know to be given by $\binom{k+r-1}{r}$ which gives us our desired binomial coefficients. ☐

We also see that this method gives us an alternative proof of the binomial coefficients for $f_d^{(a)}(n)$, for if we have $\alpha = n - (\lambda - 1)$ then we have $\frac{\alpha-a}{d+1} = \frac{n-a-(\lambda-1)}{d+1} + 1$ choices and we must pick $\lambda - 1$ pieces thus we get $\binom{\frac{n-a-(\lambda-1)}{d+1}+1+(\lambda-1)-1}{\lambda-1} = \binom{\frac{n-a+d(\lambda-1)}{d+1}}{\lambda-1}$.

We now count the number of possible arm lengths for partitions counted by $\ell_d^{(a)}(n)$.

Proposition 4.4. *For positive integers $d, a < \frac{d+3}{2}, n \geq a$, the number of possible arm lengths for partitions counted by $\ell_d^{(a)}(n)$ is given by $\left\lfloor \frac{n-a}{d+3} \right\rfloor + \left\lfloor \frac{n+a}{d+3} \right\rfloor + 1$. More specifically, the number of arm lengths $\equiv a \pmod{d+3}$ is given by $\left\lfloor \frac{n-a}{d+3} \right\rfloor + 1$, and the number of arm lengths $\equiv -a \pmod{d+3}$ is given by $\left\lfloor \frac{n+a}{d+3} \right\rfloor$.*

Proof. We start by counting the number of possible arm lengths $\equiv a \pmod{d+3}$. In this case, the possible values for α are

$$a, a + (d + 3), a + 2(d + 3), \ldots, a + s(d + 3),$$

where $a+s(d+3) \leq n$. Solving this inequality for s, we have $s \leq \left\lfloor \frac{n-a}{d+3} \right\rfloor$. And since $a+0(d+3)$ is also a possible value for α, we add one to this value to get a total of $\left\lfloor \frac{n-a}{d+3} \right\rfloor + 1$ possibilities.

Next, we count the number of possible arm lengths $\equiv -a \pmod{d+3}$. In this case, the possible values for α are

$$-a+(d+3), -a+2(d+3), ..., -a+t(d+3),$$

where $-a+t(d+3) \leq n$. Solving this inequality for t, we have $t \leq \left\lfloor \frac{n+a}{d+3} \right\rfloor$. Combining these two cases, we get a total of $\left\lfloor \frac{n-a}{d+3} \right\rfloor + \left\lfloor \frac{n+a}{d+3} \right\rfloor + 1$ possibilities. $\square$

This leads us to the following corollary.

Corollary 4.5. *For positive integers $d, a < \frac{d+3}{2}, n \geq a$, the number of possible arm lengths for $\ell_d^{(a)}(n)$ is given by the number of possible arm lengths for partitions counted by $f_{d+2}^{(a)}(n)$ plus the number of possible arm lengths for partitions counted by $f_{d+2}^{(d+3-a)}(n)$.*

Proof. By Proposition 4.4, we know the number of possible arm lengths for partitions counted by $f_d^{(a)}(n)$ is given by $\left\lfloor \frac{n-a}{d+1} \right\rfloor + 1$. Therefore the number of possible arm lengths for partitions counted by $f_{d+2}^{(a)}(n)$ is given by $\left\lfloor \frac{n-a}{d+3} \right\rfloor + 1$.

Similarly, the number of possible arm lengths for partitions counted by $f_{d+2}^{(d+3-a)}(n)$ is given by

$$\left\lfloor \frac{n-(d+3-a)}{d+3} \right\rfloor + 1 = \left\lfloor \frac{n-d-3+a}{d+3} \right\rfloor + 1 = \left\lfloor \frac{n+a}{d+3} - 1 \right\rfloor + 1 = \left\lfloor \frac{n+a}{d+3} \right\rfloor.$$

Adding these two expressions together gives us $\left\lfloor \frac{n-a}{d+3} \right\rfloor + \left\lfloor \frac{n+a}{d+3} \right\rfloor + 1$, which is the total number of possible arm lengths for partitions counted by $\ell_d^{(a)}(n)$. $\square$

We now can write $f_d^{(a)}(n)$ and $\ell_d^{(a)}(n)$ in terms of sums over all possible arm and leg length pairs which we know are given by binomial coefficients. Thus the following formulas are immediate from Theorem 1.8 and Proposition 4.3.

Corollary 4.6. *We have the following formulas for $\ell_d^{(a)}(n)$ and $f_d^{(a)}(n)$,*

$$f_d^{(a)}(n) = \sum_{k=0}^{\left\lfloor \frac{n-a}{d+1} \right\rfloor} \binom{k+n-(a+k(d+1))}{n-(a+k(d+1))},$$

$$\ell_d^{(a)}(n) = \sum_{k=0}^{\left\lfloor \frac{n-a}{d+3} \right\rfloor} \binom{2k+n-(a+k(d+3))}{n-(a+k(d+3))} + \sum_{k=1}^{\left\lfloor \frac{n+a}{d+3} \right\rfloor} \binom{2k+(n-(-a+k(d+3)))-1}{n-(-a+k(d+3))}.$$

In order to prove Theorem 1.10, we will compare these sums of binomial coefficients. We must first prove that the number of terms in the sum of $\ell_d^{(a)}(n)$ is greater than the number of terms in the sum of $f_d^{(a)}(n)$.

Lemma 4.7. *For $1 \leq d$, $a < \frac{d+3}{2}$, and for all n, the number of possible arm lengths for $\ell_d^{(a)}(n)$ is at least the number of arm lengths for $f_d^{(a)}(n)$.*

Proof. We want to prove

$$\left\lfloor \frac{n-a}{d+1} \right\rfloor + 1 \leq \left\lfloor \frac{n-a}{d+3} \right\rfloor + \left\lfloor \frac{n+a}{d+3} \right\rfloor + 1.$$

First, we shall consider $n \leq d+1$. For these n we see that $\left\lfloor \frac{n-a}{d+1} \right\rfloor = 0$ thus the inequality holds. We also see that for $n = d+2$ we also get the inequality as $\left\lfloor \frac{n-a}{d+1} \right\rfloor = 0$ unless $a = 1$, but if so then $\left\lfloor \frac{n+a}{d+3} \right\rfloor = 1$ as well so the inequality remains. Now for general a we get

$$\begin{aligned} \frac{n-a}{d+1} \leq \frac{2n}{d+3} - 1 &\iff \\ (n-a)(d+3) \leq 2n(d+1) - (d+3)(d+1) &\iff \\ n(-d+1) \leq (a-d-1)(d+3) &\iff \\ n \geq \frac{(d+1-a)(d+3)}{d-1}&. \end{aligned}$$

Thus we consider when $\frac{(d+1-a)(d+3)}{d-1} \leq d+2$ which occurs when

$$\begin{aligned} (d+1-a)(d+3) \leq (d-1)(d+2) &\iff \\ d^2 + 4d - ad + 3 - 3a \leq d^2 + d - 2 &\iff \\ 3d + 5 \leq a(d+3) &\iff \\ \frac{3d+5}{d+3} \leq a &\iff \\ \frac{3d+9}{d+3} - \frac{4}{d+3} = 3 - \frac{4}{d+3} \leq a&. \end{aligned}$$

Thus for $a \geq 3$ we know the inequality holds true for all $n \geq d+2$ and for all $d \geq 1$. Then we see that if $x \leq y$, then $\lfloor x \rfloor \leq \lfloor y \rfloor$ and that $\lfloor w + z \rfloor \leq \lfloor w \rfloor + \lfloor z \rfloor + 1$ so applying this to $x = \frac{n-a}{d+1}$, $y = \frac{2n}{d+3} - 1$ we get $\left\lfloor \frac{n-a}{d+1} \right\rfloor \leq \left\lfloor \frac{2n}{d+3} - 1 \right\rfloor = \left\lfloor \frac{2n}{d+3} \right\rfloor - 1$. Then setting $w = \frac{n-a}{d+3}$, $z = \frac{n+a}{d+3}$ we get $\left\lfloor \frac{2n}{d+3} \right\rfloor - 1 \leq \left\lfloor \frac{n-a}{d+3} \right\rfloor + \left\lfloor \frac{n+a}{d+3} \right\rfloor + 1 - 1$ giving us

$$\left\lfloor \frac{n-a}{d+1} \right\rfloor \leq \left\lfloor \frac{n-a}{d+3} \right\rfloor + \left\lfloor \frac{n+a}{d+3} \right\rfloor.$$

Now consider $a = 2$. We see that we want to show

$$\frac{n-2}{d+1} \leq \frac{2n}{d+3} - 1.$$

So following the same manipulations as above, we get

$$n \geq \frac{(d+1-2)(d+3)}{d-1} = d+3.$$

Thus since we already know the inequality is true for $a = 2$, $n \leq d+2$ we are done as this covers the rest of the cases for $a = 2$. Now if we repeat the argument for $a = 1$, we end up with

$$n \geq \frac{(d+1-1)(d+3)}{d-1} = \frac{d(d+3)}{d-1}.$$

Where the inequality holds, leaving us only $d+3 \leq n \leq \frac{d(d+3)}{d-1}$ as the remaining cases. We consider values for $f(n) = \left\lfloor \frac{n-1}{d+1} \right\rfloor$. We have the following ranges for n,

$$d+3 \leq n \leq 2d+2, \tag{4}$$

$$2d+3 \leq n \leq 3d+3, \tag{5}$$

$$3d+4 \leq n \leq 4d+4. \tag{6}$$

So we consider n within these three ranges and check that the inequality holds. For (4) we see that $\left\lfloor \frac{n+1}{d+3} \right\rfloor \geq 1$ since $d+3 \leq n \leq 2d+2$ and $1 = \left\lfloor \frac{n-1}{d+1} \right\rfloor$ so the original inequality holds. In the case of (5) we see that $2 = \left\lfloor \frac{n-1}{d+1} \right\rfloor$. We also see that $\left\lfloor \frac{n+1}{d+3} \right\rfloor \geq 1$ and $\left\lfloor \frac{n-1}{d+3} \right\rfloor \geq 1$. since $2d+3-1 = 2d+2 \geq d+3$ for $d \geq 1$ and thus $2d+3+1 \geq d+3$. Thus $\left\lfloor \frac{n+1}{d+3} \right\rfloor + \left\lfloor \frac{n-1}{d+3} \right\rfloor \geq 2 = \left\lfloor \frac{n-1}{d+1} \right\rfloor$ so the inequality holds for (5). For (6) we see that $\left\lfloor \frac{n+1}{d+3} \right\rfloor \geq 2$ since $3d+5 \geq 2d+6$ for $d \geq 1$ and

$$\left\lfloor \frac{n-1}{d+3} \right\rfloor \geq 1,$$

as the n are larger than in 5. We also know that $3 = \left\lfloor \frac{n-1}{d+1} \right\rfloor$ since $3d+4 \leq n \leq 4d+4$. So we know $\left\lfloor \frac{n+1}{d+3} \right\rfloor + \left\lfloor \frac{n-1}{d+3} \right\rfloor \geq 3 = \left\lfloor \frac{n-1}{d+1} \right\rfloor$ for all n in (6). Thus for all n such that $d+3 \leq n \leq 4d+4$ we are done. Now we compare

$$\frac{d(d+3)}{d-1} \leq 4d+4 \iff$$
$$d^2+3d \leq 4d^2-4 \iff$$
$$0 \leq 3d^2-3d,$$

which is true for all $d \geq 2$. Since we divided through by $d-1$ our previous work cannot be applied to $d=1$, but we have finished all other cases. Thus the only case left is $a=1, d=1$. So we want to show

$$\left\lfloor \frac{n-1}{2} \right\rfloor \leq \left\lfloor \frac{n-1}{4} \right\rfloor + \left\lfloor \frac{n+1}{4} \right\rfloor.$$

Consider the possible cases for n based on it's congruence $\pmod 4$.
If $n = 4k+1$, then $\left\lfloor \frac{n-1}{2} \right\rfloor = 2k$, $\left\lfloor \frac{n-1}{4} \right\rfloor = k$ and $\left\lfloor \frac{n+1}{4} \right\rfloor = k$ thus the inequality holds.
If $n = 4k+2$, then $\left\lfloor \frac{n-1}{2} \right\rfloor = 2k$, $\left\lfloor \frac{n-1}{4} \right\rfloor = k$ and $\left\lfloor \frac{n+1}{4} \right\rfloor = k$ thus the inequality holds.
If $n = 4k+3$, then $\left\lfloor \frac{n-1}{2} \right\rfloor = 2k+1$, $\left\lfloor \frac{n-1}{4} \right\rfloor = k$ and $\left\lfloor \frac{n+1}{4} \right\rfloor = k+1$ thus the inequality holds.
If $n = 4k+4$, then $\left\lfloor \frac{n-1}{2} \right\rfloor = 2k+1$, $\left\lfloor \frac{n-1}{4} \right\rfloor = k$ and $\left\lfloor \frac{n+1}{4} \right\rfloor = k+1$ thus the inequality holds.
Thus the inequality holds in all cases and we can conclude that for $a \leq n$, $1 \leq a \leq \frac{d+3}{2}$, and $1 \leq d$,

$$\left\lfloor \frac{n-a}{d+1} \right\rfloor + 1 \leq \left\lfloor \frac{n-a}{d+3} \right\rfloor + \left\lfloor \frac{n+a}{d+3} \right\rfloor + 1.$$

□

We also need another lemma in order to compare binomial coefficients.

Lemma 4.8. *For positive integers x, y, z with $y \leq z$,*

$$\binom{x+y}{y} \leq \binom{x+z}{z}.$$

Proof. We see that $\binom{x+y}{y} = \frac{(x+y)!}{x!y!}$ and $\binom{x+z}{z} = \frac{(x+z)!}{x!z!}$ thus all we have to show is that $\frac{(x+y)!}{y!} \leq \frac{(x+z)!}{z!}$. We see that

$$\frac{(x+z)!}{z!} = \left(\frac{x+z}{z}\right)\left(\frac{x+z-1}{z-1}\right)\dots\left(\frac{x+y+1}{y+1}\right)\frac{(x+y)!}{y!}.$$

Thus $\binom{x+y}{y} \leq \binom{x+z}{z}$. □

We now have the necessary tools to prove 1.10.

4.1. Proof of Theorem 1.10.

First Proof of Theorem 1.10. Because $f_d^{(a)}(n) = h_d^{(a)}(n)$, it will suffice to show $f_d^{(a)}(n) \leq \ell_d^{(a)}(n)$. We start by considering the series representations for $f_d^{(a)}(n)$ and $\ell_d^{(a)}(n)$ given in Corollary 4.6. By expanding and combining like terms, these simplify to

$$f_d^{(a)}(n) = \sum_{k=0}^{\left\lfloor \frac{n-a}{d+1} \right\rfloor} \binom{n-a-kd}{n-a-kd-k}$$

and

$$\ell_d^{(a)}(n) = \sum_{k=0}^{\left\lfloor \frac{n-a}{d+3} \right\rfloor} \binom{n-a-kd-k}{n-a-kd-3k} + \sum_{k=1}^{\left\lfloor \frac{n+a}{d+3} \right\rfloor} \binom{n+a-kd-k-1}{n+a-kd-3k}.$$

To better match the shape of the sum formula for $\ell_d^{(a)}(n)$, we split the sum formula for $f_d^{(a)}(n)$ into two sums: one for $k = 2j$ (even k) and one for $k = 2j-1$ (odd k). Then we have

$$f_d^{(a)}(n) = \sum_{j=0}^{\left\lfloor \frac{\left\lfloor \frac{n-a}{d+1} \right\rfloor}{2} \right\rfloor} \binom{n-a-2jd}{n-a-2jd-2j} + \sum_{j=1}^{\left\lfloor \frac{\left\lfloor \frac{n-a}{d+1} \right\rfloor+1}{2} \right\rfloor} \binom{n-a-(2j-1)d}{n-a-(2j-1)d-(2j-1)}.$$

Now we wish to show

$$\sum_{j=0}^{\left\lfloor \frac{n-a}{d+3} \right\rfloor} \binom{n-a-jd-j}{n-a-jd-3j} \geq \sum_{j=0}^{\left\lfloor \frac{\left\lfloor \frac{n-a}{d+1} \right\rfloor}{2} \right\rfloor} \binom{n-a-2jd}{n-a-2jd-2j}$$

and

$$\sum_{j=1}^{\left\lfloor \frac{n+a}{d+3} \right\rfloor} \binom{n+a-jd-j-1}{n+a-jd-3j} \geq \sum_{j=1}^{\left\lfloor \frac{\left\lfloor \frac{n+a}{d+1} \right\rfloor+1}{2} \right\rfloor} \binom{n-a-(2j-1)d}{n-a-(2j-1)d-(2j-1)}.$$

We will start with the even case, in which we want to show that

$$\left\lfloor \frac{\left\lfloor \frac{n-a}{d+1} \right\rfloor}{2} \right\rfloor \leq \left\lfloor \frac{n-a}{d+3} \right\rfloor.$$

Using the fact that $x \geq y$ implies $\lfloor x \rfloor \geq \lfloor y \rfloor$, it will suffice to show that $\frac{\lfloor \frac{n-a}{d+1} \rfloor}{2} \leq \frac{n-a}{d+3}$, or $\lfloor \frac{n-a}{d+1} \rfloor \leq 2(\frac{n-a}{d+3})$. Now since $\lfloor \frac{n-a}{d+1} \rfloor \leq \frac{n-a}{d+1}$, we may prove that $\frac{n-a}{d+1} \leq \frac{2(n-a)}{d+3}$. This simplifies to $d+3 \leq 2(d+1)$, which is true for any $d \geq 1$. Therefore we can conclude that $\left\lfloor \frac{\lfloor \frac{n-a}{d+1} \rfloor}{2} \right\rfloor \leq \lfloor \frac{n-a}{d+3} \rfloor$ for any $d \geq 2$.

Now we show that for every $j \geq 0$,

$$\binom{n-a-jd-j}{n-a-jd-3j} \geq \binom{n-a-2jd}{n-a-2jd-2j}.$$

Notice that on both sides of the inequality, the top term in the binomial coefficient is exactly $2j$ greater than the bottom term. Furthermore, comparing the top terms on either side, we see that $n-a-jd-j \geq n-a-2jd$ for any $d \geq 1$. Then by applying Lemma 4.8, we can see that our inequality holds.

Next, we use a similar argument to prove the odd case. We start by showing that

$$\left\lfloor \frac{\left\lfloor \frac{n-a}{d+1} + 1 \right\rfloor}{2} \right\rfloor \leq \left\lfloor \frac{n+a}{d+3} \right\rfloor. \tag{7}$$

Once again using the fact that $x \geq y$ implies $\lfloor x \rfloor \geq \lfloor y \rfloor$, we instead show that $\frac{\lfloor \frac{n-a}{d+1}+1 \rfloor}{2} \leq \frac{n+a}{d+3}$, or $\lfloor \frac{n-a}{d+1} + 1 \rfloor \leq 2(\frac{n+a}{d+3})$. Now since $\lfloor \frac{n-a}{d+1} + 1 \rfloor \leq \frac{n-a}{d+1} + 1$, we will show $\frac{n-a}{d+1} \leq 2(\frac{n+a}{d+3}) - 1$.

This simplifies to the inequality

$$\frac{-3ad - 5a + d + 1}{d-1} \leq n.$$

We first consider the case where $d \geq 2$. Since we know n is positive, it will suffice to show that $-3ad - 5a + d + 1 \leq 0$. Solving for a, we get $a \geq \frac{d+1}{3d+5}$, which is true for any positive integer a because $\frac{d+1}{3d+5} < 1$. To prove the case where $d = 1$, we plug $d = 1$ directly into (7) to see that the inequality clearly holds.

Now, we show that for any $j \geq 1$,

$$\binom{n+a-jd-j-1}{n+a-jd-3j} \geq \binom{n-a-(2j-1)d}{n-a-(2j-1)d-(2j-1)}.$$

In this case, we see on both sides of the inequality that the top term in the binomial coefficient is exactly $2j-1$ greater than the bottom term. Now we just have to compare the top terms on either side of the inequality. To do this, we subtract the top term on the right hand side from the top term on the left hand side to get that for any $j \geq 1$,

$$n + a - jd - j - 1 - (n - a - (2j-1)d) = 2a + jd - j - d - 1 = 2a + (j-1)(d+1) \geq 0.$$

Once again applying Lemma 4.8, this completes the proof. □

We now give our second proof.

Second Proof of Theorem 1.10. We will prove that $h_d^{(a)}(n) = f_d^{(a)}(n) \leq \ell_d^{(a)}(n)$ by considering each function as the sum of binomial coefficients in Corollary 4.6. From Theorem 1.8 and Proposition 4.3 we see that both coefficients come from the formula $\binom{s+r-1}{r}$ where s is the amount of possible lengths for each of the pieces in the leg or a partition, and r is the length of the leg in the partition. We see that when $\alpha = a$, both binomial coefficients will have $s = 1$ and $r = \lambda - 1 = n - a$ so they shall be equal. If we arrange the sequences of possible α values for $f_d^{(a)}(n)$ we get $a, a+(d+1), a+2(d+1), a+3(d+1)...$ with respective s values of $1, 2, 3, 4, ...$ and if we do the same for $\ell_d^{(a)}(n)$ we get $a, -a+(d+3), a+(d+3), -a+2(d+3), ...$ with the same respective s values of $1, 2, 3, 4, ...$ For a given s, we denote $\lambda_{s,f}$ to be the number of pieces in the partitions counted by $f_d^{(a)}(n)$ with s possible lengths and respectively we define $\lambda_{s,\ell}$. We also define $\alpha_{s,f}$ and $\alpha_{s,\ell}$ to be the lengths of the largest pieces of the partitions counted by $f_d^{(a)}(n)$ and $\ell_d^{(a)}(n)$ with s possible lengths. We know by Lemma 4.7 that the sequence of possible arm lengths for $\ell_d^{(a)}(n)$ is longer than or equal to the length of the sequence of arm lengths for $f_d^{(a)}(n)$. Thus we will show that for fixed s,

$$\binom{s-1+(\lambda_{s,f}-1)}{\lambda_{s,f}-1} \leq \binom{s-1+(\lambda_{s,\ell}-1)}{\lambda_{s,\ell}-1}$$

which will prove that for all the terms in the sum of $f_d^{(a)}(n)$ have a corresponding larger binomial coefficient in the sum of $\ell_d^{(a)}(n)$ by Lemma 4.8. By Lemma 4.7 since the sequence of binomial coefficients in $\ell_d^{(a)}(n)$ is longer than the sequence in $f_d^{(a)}(n)$ then if we prove the term-wise inequality of the coefficients we will be done. For $f_d^{(a)}(n)$, $s = \frac{\alpha-a}{d+1} + 1$ so $\alpha_{s,f} = a + (s-1)(d+1)$ and for $\ell_d^{(a)}(n)$ we see that for $\alpha \equiv a \pmod{d+3}$, $s = 2\left(\frac{\alpha-a}{d+3}\right) + 1$ and for $\alpha \equiv -a \pmod{d+3}$, $s = 2\left(\frac{\alpha+a}{d+3}\right)$ so $\alpha_{s,\ell} = a + \left(\frac{s-1}{2}\right)(d+3)$ for odd s and $\alpha_{s,\ell} = -a + \frac{s}{2}(d+3)$ for even s. Thus in order to show $\lambda_{s,f} \leq \lambda_{s,\ell}$ for fixed s we shall show that $\alpha_{s,f} \geq \alpha_{s,\ell}$ for all s. So we want to show that for $a \leq \frac{d+3}{2}, d \geq 1, s \geq 1$, s odd,

$$a + (s-1)(d+1) - \left(a + \left(\frac{s-1}{2}\right)(d+3)\right) \geq 0,$$

and for $a < \frac{d+3}{2}, d \geq 2, s \geq 1$, s even,

$$a + (s-1)(d+1) - \left(-a + \frac{s}{2}(d+3)\right) \geq 0.$$

For the first inequality, we see the left hand side simplifies to

$$= ds - d + s - 1 - \frac{ds}{2} + \frac{d}{2} - \frac{3s}{2} + \frac{3}{2} = \frac{ds}{2} - \frac{d}{2} - \frac{s}{2} + \frac{1}{2}$$
$$= \frac{ds}{2} - \frac{d}{2} - \frac{s}{2} + \frac{1}{2} = \frac{(s-1)(d-1)}{2}$$

which is clearly non-negative for $d \geq 1, s \geq 1$.
For the second inequality we see the left hand side simplify to

$$= a + (s-1)(d+1) - \left(-a + \frac{s}{2}(d+3)\right) = 2a + ds - d + s - 1 - \frac{ds}{2} - \frac{3s}{2}$$

$$= 2a + \frac{ds}{2} - d - \frac{s}{2} - 1 \geq \frac{ds}{2} - d - \frac{s}{2} + 1$$

since $a \geq 1$ and we see that

$$\frac{ds}{2} - d - \frac{s}{2} + 1 = (\frac{s}{2} - 1)(d+1)$$

is clearly non-negative for $s \geq 2, d \geq 1$. □

5. Inequality Chains

Proof of Theorem 1.11. First we analyze the inequalities for $h_d^{(a)}(n)$. The first two chains are given by natural inclusions of $h_{d+1}^{(a)}(n)$ into $h_d^{(a)}(n)$ and by $h_d^{(a+1)}(n)$ into $h_d^{(a)}(n)$. The third inequality is achieved by an injection of $h_d^{(a)}(n)$ into $h_d^{(a)}(n+1)$ given by adjoining an extra part to the largest part of any partition counted by $h_d^{(a)}(n)$, transforming it into a partition counted by $h_d^{(a)}(n+1)$. Now we look to the inequalities for $\ell_d^{(a)}(n)$. We have an injection from the partitions counted by $\ell_d^{(a)}(n)$ to $\ell_d^{(a)}(n+1)$. For any partition all we do is add an extra row of size a to the bottom of a partition which gives us our injection.

Lastly we prove that $\ell_d^{(a)}(n) \geq \ell_{d+1}^{(a)}(n)$. We know from 4.4 that the number of possible arm lengths for partitions in $\ell_d^{(a)}(n)$ is given by

$$\left\lfloor \frac{n-a}{d+3} \right\rfloor + \left\lfloor \frac{n+a}{d+3} \right\rfloor + 1.$$

Therefore, the number of possible arm lengths for partitions in $\ell_{d+1}^{(a)}(n)$ must be

$$\left\lfloor \frac{n-a}{d+4} \right\rfloor + \left\lfloor \frac{n+a}{d+4} \right\rfloor + 1,$$

which is clearly less than or equal to the number of possible arm lengths for $\ell_d^{(a)}(n)$.

Now recall the sum formula for $\ell_d^{(a)}(n)$ from Corollary 4.6. Since we just showed that the series for $\ell_d^{(a)}(n)$ has at least as many terms as $\ell_{d+1}^{(a)}(n)$, we may compare the series term-by-term to prove our result. We will start by showing

$$\binom{2k+n-(a+k(d+3))}{n-(a+k(d+3)} \geq \binom{2k+n-(a+k(d+4))}{n-(a+k(d+4))}$$

which simplifies to

$$\binom{n-a-kd-k}{n-a-kd-3k} \geq \binom{n-a-kd-2k}{n-a-kd-4k}.$$

Notice that both of these binomial coefficients have the property that the top term is exactly $2k$ greater than the bottom term, and therefore it will suffice to compare only the top terms by Lemma 4.8. Comparing the top terms, we can clearly see that $n-a-kd-k \geq n-a-kd-2k$. Therefore

$$\sum_{k=0}^{\lfloor \frac{n-a}{d+3} \rfloor} \binom{2k+n-(a+k(d+3))}{n-(a+k(d+3))} \geq \sum_{k=0}^{\lfloor \frac{n-a}{d+4} \rfloor} \binom{2k+n-(a+k(d+4))}{n-(a+k(d+4))}.$$

Now, we will show

$$\binom{2k+n-(a+k(d+3))-1}{n-(-a+k(d+3))} \geq \binom{2k+n-(a+k(d+4))-1}{n-(-a+k(d+4))},$$

which simplifies to

$$\binom{n+a-kd-k-1}{n+a-kd-3k} \geq \binom{n+a-kd-2k-1}{n+a-kd-4k}.$$

Notice that both binomial coefficients have the property that the top term is exactly $2j-1$ greater than the bottom term, so again we may compare just the top terms by Lemma 4.8. We can clearly see that $n+a-kd-k-1 \geq n+a-kd-2k-1$, so therefore

$$\sum_{k=1}^{\lfloor \frac{n+a}{d+3} \rfloor} \binom{2k+n-(a+k(d+3))-1}{n-(-a+k(d+3))} \geq \sum_{k=1}^{\lfloor \frac{n+a}{d+4} \rfloor} \binom{2k+n-(a+k(d+4))-1}{n-(-a+k(d+4))}.$$

□

The same results for $h_d^{(a)}(n)$ are true for $q_d^{(a)}(n)$ but the fact that $h_d^{(a)}(n) = f_d^{(a)}(n)$ extends results to $f_d^{(a)}(n)$ which do not extend to $Q_d^{(a)}(n)$. In fact, all these results fail for $Q_d^{(a)}(n)$ and can all fail at the same time. Consider $d=6, a=2, n=18$ and we see that

$$\begin{aligned} 4 = Q_6^{(2)}(18) &\geq Q_6^{(2)}(19) = 2 \\ 4 = Q_6^{(2)}(18) &\leq Q_6^{(3)}(18) = 7 \\ 4 = Q_6^{(2)}(18) &\leq Q_7^{(2)}(18) = 5. \end{aligned}$$

6. Further Questions for Fixed Perimeter Partitions

Kang and Kim [8] generalized $Q_d^{(a)}$ so that instead of parts being congruent to $\pm a \pmod{d+3}$, they could be congruent to any two congruence classes $a, b \pmod{d+3}$. This leads us to the following definition.

Definition 6.1. *For positive integers a, b, d, and n,*

$$\begin{aligned} \ell_d^{(a,b)}(n) &= r(n | \textit{with parts } \equiv a, b \pmod{d+3}) \\ &= \sum_{k=0}^{\lfloor \frac{n-a}{d+3} \rfloor} \binom{n-q-kd-k}{n-a-kd-3k} + \sum_{k=1}^{\lfloor \frac{n-b}{d+3} \rfloor} \binom{n-b-kd-k-1}{n-b-kd-3k}. \end{aligned}$$

We see that by replacing every piece congruent to $-a$ with a piece congruent to b, we can achieve a bijection between $\ell_d^{(a)}(n)$ and $\ell_d^{(a,b)}(n+(b-(-a+d+3)))$. This bijection sends a part $-a+k(d+3)$ to $b+k(d+3)$. Thus we see that

$$\ell_d^{(a)}(n) = \ell_d^{(a,b)}(n+(b-(-a+d+3))).$$

While inequality chains have been found for $\ell_d^{(a)}(n)$ while only varying d or n, we were unable to find a proof for an inequality chain for varying a with fixed d, n. Computational evidence suggests that for small values of n, $\ell_d^{(a)}(n) \geq \ell_d^{(a+1)}(n)$, but only for a small finite number of values for each a that we checked. It seems that $\ell_d^{(a)}(n) \leq \ell_d^{(a+1)}(n)$ for all but finite n for a given a.

There are also further open questions about analogues to Kang and Kim [8] relating to comparisons between $h_p^{(p_1)}(n)$ and $\ell_m^{(m_1)}(n)$ with different parameter values along with generalizations of $\ell_m^{(m_1)}(n)$ to $\ell_m^{(m_1,m_2)}(n)$.

7. Approaches for Remaining Cases of Kang and Park's Conjecture

For Kang and Park's conjecture, the only remaining cases are $d = 3, 4, 5$ and 7. While we have not found proofs for these cases, we will survey the current literature and share our attempted approaches.

Previous work has provided a number of lemmas such as [4, 2.4, 2.5, 5.2, 6.1]. Each of these lemmas can be applied to some or all of the remaining cases, or slightly modified to do so, but the resulting inequalities fail to act as proper intermediaries in all cases. There are asymptotic bounds for $d = 4, 5$, and 7 but the bounds were too high for Sturman and Swisher [11] to check all remaining cases with code. No asymptotics exist for $d = 3$ since they were not needed for Alder's conjecture so did not appear in the work of [1] and thus were not considered by Sturman and Swisher.

For $d = 3$, computational evidence suggests that $q_3^{(2)}(n) \geq Q_3^{(1,-)}(n)$ for all $n \geq 2$, and $Q_3^{(1,-)}(n) \geq Q_3^{(2,-)}(n)$ for all $n \geq 17$. Furthermore, q-series manipulations following from Schur's Theorem show allow us to write $Q_3^{(1,-)}(n) = D_3^{(1)}(n) - D_3^{(1)}(n-5)$, where we define

$$D_t^{(1)}(n) = p(n|\text{distinct parts} \equiv \pm 1 \pmod{t}).$$

Therefore, it would suffice to show $q_3^2(n) \geq D_3^{(1)}(n) - D_3^{(1)}(n-5)$ and $Q_3^{(1,-)}(n) \geq Q_3^{(2,-)}(n)$. A possible approach to proving the former could be finding an injection that maps partitions in $D_3^{(1)}(n)$ to partitions in $q_3^{(2)}(n) \cup D_3^{(1)}(n-5)$, but we were not able to find such a mapping. The latter inequality could potentially be proved using q-series manipulations.

Since computational evidence suggests that

$$q_3^{(2)}(n) \geq D_3^{(1)}(n) + D_3^{(1)}(n-5) \geq Q_3^{(2,-)}(n),$$

we looked to see if we could find any similar intermediate functions for $q_4^{(2)}(n)$ and $Q_4^{(2,-)}(n)$. We found that computational evidence suggests that

$$q_4^{(2)}(n) \geq D_4^{(1)}(n) \geq Q_4^{(2,-)}(n),$$

which is a simpler intermediate function as there is no need to subtract another set. We were unable to find injections for this intermediate step.

For $d = 7$, computational evidence suggests that

$$q_7^{(2)}(2n) \geq q_3^{(1)}(n) \geq Q_2^{(1,-)}(n) = Q_7^{(2,-)}(2n).$$

By modifying Lemma 2.4 from [4], it may be possible to find an injection from $q_3^{(1)}(n)$ into $q_7^{(2)}(2n)$.

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