OSU Department of Mathematics
Qualifying Examination
Fall 2023

Linear Algebra

Instructions:

- Do **any three** of the four problems.
- Use separate sheets of paper for each problem. Clearly indicate the problem and page number (if several pages are used for a solution) on the top of the page.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have **four** hours to complete this examination.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination:
  1. Use the problem selection sheet to indicate your identification number and the three problems which you wish to be graded.
  2. Arrange your solutions according to the problem order with the problem selection sheet on top and any scratch-work on the bottom.
  3. Submit the exam: place your solutions together with the selection sheet and scratch paper, in the order arranged as above, into the envelope in which you received the exam and submit it to the proctor.

Exam continues on next page ...

Common notations:

- $\mathbb{R}^{n\times m}$ denotes the space of real $n \times m$ matrices with n rows and m columns
- I_n denotes the n-dimensional identity matrix in $\mathbb{R}^{n\times n}$
- A^t denotes the transpose of matrix A

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Problems:

1. (10 pts) Let B be the real $n \times n$ matrix with all entries equal to 1. Find a real diagonal matrix similar to B or prove that none exists.

2. **a.** (5 pts) Assuming basic facts about normal operators, prove that if T is a normal operator on a finite dimensional complex inner product space V, then any eigenvectors of T corresponding to distinct eigenvalues must be orthogonal.

   b. (5 pts) Give an explicit example of a diagonalizable operator on $\mathbb{C}^n$ for some n for which there exist nonorthogonal eigenvectors corresponding to distinct eigenvalues. Justify your example.

3. For integer $n, m \geq 1$, consider the subset $\mathcal{S}$ of $\mathbb{R}^{n \times m}$ consisting of all matrices for which the sum of all even-indexed column vectors equals the sum of all odd-indexed column vectors. That is, $\mathcal{S}$ consists of all matrices $A \in \mathbb{R}^{n \times m}$ whose column vectors $\mathbf{r}_1, \ldots, \mathbf{r}_m$ satisfy

$$\sum_{i \text{ even}} \mathbf{r}_i = \sum_{i \text{ odd}} \mathbf{r}_i.$$

   a. (2 pts) Show that $\mathcal{S}$ is a vector subspace of $\mathbb{R}^{n \times m}$.

   b. (4 pts) Show that for any $A \in \mathbb{R}^{n \times n}$ and $B \in \mathcal{S}$, their product $AB \in \mathcal{S}$.

   c. (4 pts) For $A \in \mathcal{S}$ show that matrix $I_m + A^t A$ is invertible and find its least eigenvalue.

4. (10 pts) For a given real $\lambda \neq 0$, consider $n \times n$ matrix

$$L = \begin{pmatrix} 1 & \lambda & 0 & \cdots & 0 \\ 0 & 1 & \lambda & \cdots & \vdots \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \lambda \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

   Find a Jordan canonical basis for L.