GENERALIZED PARTITION IDENTITIES AND FIXED PERIMETER ANALOGUES

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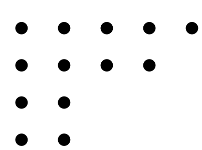
Abstract. Euler's classic partition identity states that the number of partitions of n into odd parts equals the number of partitions of n into distinct parts. We develop a new generalization of this identity, which yields a previous generalization of Franklin as a special case, and prove an accompanying Beck-type companion identity.

Strikingly, in 2016, Straub proved that Euler's identity holds true for partitions with largest hook (perimeter) n. This inspired further study of the relationship between classical partitions and fixed perimeter partitions. Motivated by recent findings in this area, we develop fixed perimeter analogues of several standard partition results as well as a Beck-type companion identity in the fixed perimeter setting. We further use combinatorial methods to prove analogues of various results related to parity in the fixed perimeter setting.

1. Introduction and Statement of Results

A *partition* of a positive integer n is a finite non-increasing sequence of positive integers $\pi = (\pi_1, \pi_2, \ldots, \pi_m)$ such that $\sum_{i=1}^{m} \pi_i = n$. The π_i are referred to as *parts*. The partition function $p(n)$ counts the number of partitions of n, and $p(n \mid *)$ counts the number of partitions of n satisfying some condition $*$.

One way to visualize a partition is with a *Ferrers diagram*, where each part π_i is represented by a row of π_i dots. The rows are left-justified and arranged in non-increasing order with the largest part in the first row. For example, the following diagram represents the partition $(5, 4, 2, 2)$.



A fundamental partition theorem of Euler states that for any positive integer n, the number of partitions of n into odd parts is equal to the number of partitions of n into distinct parts. Employing standard notation, we have

$$p(n \mid \text{odd parts}) = p(n \mid \text{distinct parts}). \tag{1}$$

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If we think of odd parts as parts not divisible by 2 and distinct parts as parts that appear less than 2 times, then Glaisher's theorem [18] generalizes Euler's theorem by stating that for all nonnegative integers n and all $k \geq 2$,

$$p(n \mid \text{no part divisible by } k) = p(n \mid \text{no part appears } \geq k \text{ times}). \tag{2}$$

In 1883, Franklin [15] provided a further generalization of Euler's theorem (1). Define the functions $O_{j,k}(n)$ to count the number of partitions of n with exactly j parts divisible by k (repetitions allowed) and $D_{j,k}(n)$ to count the number of partitions of n with exactly j parts that appear at least k times. Then, Franklin's Theorem states that for all nonnegative integers n, j, and all $k \geq 2$,

$$O_{j,k}(n) = D_{j,k}(n). \tag{3}$$

Note that setting $j = 0$ in (3) gives (2) and setting $j = 0$, $k = 2$ gives (1).

We also have the following theorem [19] of Hovey et al., which can also be thought of as a generalization of Euler and Glaisher. For all nonnegative integers n, $k \geq 2$, and $b \geq 1$, Hovey et al. gives that

$$\begin{aligned} &p(n \mid \text{no part is divisible by } kb) \\ &\qquad = p(n \mid \text{no part is both divisible by } b \text{ and appears } \geq k \text{ times}). \end{aligned} \tag{4}$$

In order to write (1), (2), (3), and (4) in a more uniform notation, we introduce another parameter b to the $O_{j,k}(n)$ and $D_{j,k}(n)$ functions. Let $O_{j,k,b}(n)$ count the number of partitions of n with exactly j parts divisible by kb (repetitions allowed) and $D_{j,k,b}(n)$ count the number of partitions of n with exactly j parts are both divisible by b and appear $\geq k$ times. We also let $\mathcal{O}_{j,k,b}(n)$ and $\mathcal{D}_{j,k,b}(n)$ denote the set of all partitions counted by $O_{j,k,b}(n)$ and $D_{j,k,b}(n)$, respectively, so that

$$O_{j,k,b}(n) = |\mathcal{O}_{j,k,b}(n)| \quad \text{and} \quad D_{j,k,b}(n) = |\mathcal{D}_{j,k,b}(n)|.$$

We now state our first result.

Theorem 1.1. *For all non-negative integers n, $j \geqslant 0$, $k \geqslant 2$, and $b \geqslant 1$,*

$$O_{j,k,b}(n) = D_{j,k,b}(n).$$

In 2017, George Beck posted a conjecture about some sequences, say $a(n)$, $b(n)$, and $c(n)$, on OEIS in [25]. Let $a(n)$ count the number of partitions of n such that the set of even parts has only one element, $b(n)$ count the difference between the total number of parts in all odd partitions of n and the total number of parts in all distinct partitions of n, and $c(n)$ count the number of partitions of n in which exactly one part is repeated. Later that year, Andrews [5, Thm. 1] proved Beck's conjecture, showing that $a(n) = b(n) = c(n)$ for all $n \geq 1$. Such identities are now called Beck-type companion identities.

Also in 2017, Fu and Tang [16, Thm. 1.5] generalized the above result of Andrews as follows. For all $n \geq 0$ and $k \geq 2$, we have that

$$O_{1,k}(n) = \sum_{\pi \in \mathcal{O}_{0,k}(n)} \ell_1(\pi) - \sum_{\pi \in \mathcal{D}_{0,k}(n)} \bar{\ell}(\pi) = D_{1,k}(n),$$

where $\ell_1(\pi)$ is the number of parts $\equiv 1 (\mathrm{mod}\ k)$ in the partition π and $\bar{\ell}(\pi)$ is the number of different parts in the partition π.

Definition 1.2. *For all nonnegative integers n, j, and all integers $k \geq 2$, $b \geq 1$, define*

$$E(O_{j,k,b}(n), D_{j,k,b}(n)) = \sum_{\pi \in \mathcal{O}_{j,k,b}(n)} \ell(\pi) - \sum_{\pi \in \mathcal{D}_{j,k,b}(n)} \ell(\pi),$$

where $\ell(\pi)$ is the total number of parts in the partition π.

Ballantine and Welch [8, Thm. 3] also discovered Beck-type companion identities for Franklin's Theorem. They find that for all nonnegative integers n, j, and all $k \geq 2$,

$$\frac{1}{k-1} E(O_{j,k}(n), D_{j,k}(n)) = (j+1)O_{j+1,k}(n) - jO_{j,k}(n) = (j+1)D_{j+1,k}(n) - jD_{j,k}(n). \tag{5}$$

We state our next result, which is a Beck-type companion identity for Theorem 1.1 that generalizes (5).

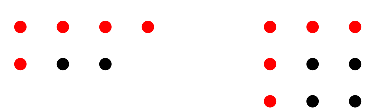
Theorem 1.3. *For all $n \in \mathbb{N}$, $k \geq 2$, and j, $b \geq 1$,*

$$\frac{1}{k-1} E(O_{j,k,b}(n), D_{j,k,b}(n)) = (j+1)O_{j+1,k,b}(n) - jO_{j,k,b}(n) = (j+1)D_{j+1,k,b}(n) - jD_{j,k,b}(n).$$

We may also count partitions with a fixed perimeter rather than a fixed size. For a partition $\pi = (\pi_1, \pi_2, \ldots, \pi_m)$, let $\alpha(\pi) := \pi_1$ be the largest part, which we also call the arm length, and let $\lambda(\pi) := m$ be the number of parts, which we also call the leg length. (Note that $\lambda(\pi) = \ell(\pi)$ as leg length and number of parts are equivalent, however, different scenarios dictate the use of one over the other.) Define the *perimeter* of a partition π to be

$$\Gamma(\pi) = \alpha(\pi) + \lambda(\pi) - 1.$$

Similarly to $p(n)$, we define $r(n)$ to count the number of partitions of perimeter n. For example, below are two partitions with perimeter 5.



In 2016, Straub [26, Thm. 1.4] discovered an analogue to Euler's identity for partitions with perimeter n. Let F_n denote the nth Fibonacci number. For all nonnegative integers n, Straub proved that

$$r(n \mid \text{odd parts}) = r(n \mid \text{distinct parts}) = F_n. \tag{6}$$

This result has motivated further investigation into whether other results for partitions of size n hold true for partitions of fixed perimeter n.

In 2018, Fu and Tang [17, Thm. 2.15] generalized (6), which was then refined in 2024 by Chen et al. [12, Thm. 1.4] as follows. Define the functions

$$\begin{aligned} h_d^{(a)}(n) &:= r(n \mid \text{ parts are d-distinct and } \geq a), \\ f_d^{(a)}(n) &:= r(n \mid \text{ parts are } \equiv a (\mathrm{mod}\ d+1)). \end{aligned} \tag{7}$$

Then for positive integers d, n, $1 \leq a \leq d+1$, Chen et al. proved that

$$h_d^{(a)}(n) = f_d^{(a)}(n). \tag{8}$$

If we define

$$FO_{j,k}(n) \coloneqq r(n \mid \text{exactly } j \text{ parts are divisible by } k),$$
$$FD_{j,k}(n) \coloneqq r(n \mid \text{exactly } j \text{ parts appear } \geq k \text{ times}),$$

then we have the following fixed perimeter result.

Theorem 1.4. *For all non-negative integers n, $j \geq 0$,*

$$FO_{j,2}(n) = FD_{j,2}(n).$$

Note that for $j = 0$, this is precisely (6), and for $j = 1$, we obtain a result of Amdeberhan et al. [1, Thm. 1.1].

We are also interested in Andrews's S-T Theorem [2, Thm. 3], which states that for sets $S = \{a_0, a_1, a_2, \ldots\}$ and $T = \{1 = b_0, b_1, b_2, \ldots\}$, with $b_i \leq a_i$, $a_{i+1} > a_i$, and $b_{i+1} > b_i$ for $i \in \mathbb{N}_0$, we have

$$p(n \mid \text{ parts in } T) \geq p(n \mid \text{ parts in } S). \tag{9}$$

For partitions with fixed perimeter n, we find that (9) holds true and we are no longer required to choose $b_0 = 1$. Our next result is a fixed-perimeter analogue to (9).

Theorem 1.5. *Let $S = \{a_0, a_1, a_2, \ldots\}$, $T = \{b_0, b_1, b_2, \ldots\}$ where $b_i \leq a_i$, $a_{i+1} > a_i$, and $b_{i+1} > b_i$ for $i \in \mathbb{N}_0$. Then*

$$r_T(n) = r(n \mid \text{ parts in } T) \geq r(n \mid \text{ parts in } S) = r_S(n).$$

We now introduce the concept of parity bias in partitions.

Definition 1.6. *Let $p_o(n)$ be the number of partitions of n with more odd parts than even parts and $p_e(n)$ be the number of partitions of n with more even parts than odd parts.*

In 2020, Kim, Kim, and Lovejoy [23, Thm. 1] prove that for all $n \neq 2$,

$$p_e(n) < p_o(n). \tag{10}$$

Letting $r_o(n)$ (resp. $r_e(n)$) denote the number of perimeter n partitions with more odd parts than even parts (resp. more even parts than odd parts), we prove an analogous inequality to (10) for partitions with fixed perimeter (noting equality holds when $n = 2$).

Theorem 1.7. *For $n \neq 2$,*

$$r_e(n) < r_o(n).$$

In addition, we find that the structure of fixed perimeter partitions easily motivates combinatorial arguments for parity bias inequalities, and we are able to develop combinatorial proofs for the fixed perimeter analogues of many other parity bias results. We find that the same combinatorial arguments can be used to justify analogous inequalities for perimeter n partitions in which all odd parts or all even parts are required to be distinct. These are described in Section 5.

We now outline the remainder of this paper. In Section 3, we provide two proofs of Theorem 1.1, one via generating functions and the other via a combinatorial bijection. We also provide two proofs of the Beck-type companion identity in Theorem 1.3, one using differentiation of generating functions and one incorporating the refinement approach of Ballantine and Welch in [8, Thm. 4]. In Section 4, we prove Theorem 1.4, Theorem 1.5, as well as other fixed perimeter analogues including a Beck-type companion identity. In Section 5, we prove Theorem 1.7 as well as a number of generalizations related to parity in the fixed perimeter setting and recursive formulas for the number of perimeter n partitions requiring even parts to be distinct or odd parts to be distinct.

2. Some Partition Results

In this section, we state several results that we consider in the fixed perimeter setting. In 2018, Fu and Tang [17, Thm. 2.15] generalized (6) as follows. First, recall the definitions of $h_d^{(a)}(n)$ and $f_d^{(a)}(n)$ given by (7). For all nonnegative integers n and d, Fu and Tang proved that

$$h_d^{(1)}(n) = f_d^{(1)}(n). \tag{11}$$

In 2024, Chen et al. [12] generalized (11) as described in (8). We define another function $\ell_d^{(a)}(n)$ in order to discuss more results of Chen et al. Let

$$\ell_d^{(a)}(n) := r(n \mid \text{parts are } \equiv \pm a (\text{mod } d+3)).$$

For positive integers d, n, and $a < \frac{d+3}{2}$, Chen et al. [12, Thm. 1.5] proved that

$$h_d^{(a)}(n) \le \ell_d^{(a)}(n). \tag{12}$$

There are also shift inequalities for $h_d^{(a)}(n)$ and $\ell_d^{(a)}(n)$ given by Chen et al. in [12, Prop. 1.6]. For positive integers d, n, and a,

$$h_d^{(a+1)}(n) \le h_d^{(a)}(n), \tag{13}$$

$$h_{d+1}^{(a)}(n) \le h_d^{(a)}(n), \tag{14}$$

$$\ell_{d+1}^{(a)}(n) \le \ell_d^{(a)}(n). \tag{15}$$

In 2020, Duncan et al. [14, Lemma 2.2] give a generalization of (9), further extending on an earlier generalization proved by Kang and Park in [21, Lemma 2.3].

Lemma 2.1 (Duncan et al. [14], 2020)**.** *Let* $S = \{a_0, a_1, a_2, \ldots\}$, $T = \{m, b_1, b_2, \ldots\}$ *such that* m *divides each* b_i *and* $b_i \le a_i$, $a_{i+1} > a_i$, $b_{i+1} > b_i$ *for* $i \in \mathbb{N}$. *Then*

$$p(n \mid \textit{ parts in } T) \ge p(n \mid \textit{ parts in } S).$$

We note that our fixed perimeter analogue Theorem 1.5 also covers the analogue of this theorem.

There is also a fixed perimeter Beck-type identity from Amdeberhan et al. [1, Cor. 1.8] which states that for all non-negative integers n,

$$E(FO_{0,2}(n), FD_{0,2}(n)) = \text{the number of partitions in } \mathcal{FO}_{1,2}(n) \text{ with no part equal to 1.}$$

We now define

$$q_d^{(a)}(n) \coloneqq p(n \mid \text{ parts are } d\text{-distinct and } \geq a),$$
$$Q_m^{(m_1,m_2)}(n) \coloneqq p(n \mid \text{parts} \equiv m_1, m_2 \pmod{m}).$$

Kang and Kim [20, Thm. 1.1] give the following theorem.

Theorem 2.2 (Kang, Kim [20], 2021). *For integers $0 \leqslant m_1 < m_2 < m$ and $a, d \geqslant 1$, let $\alpha_d \in (0,1)$ be a root of $x^d + x - 1$ and define*

$$A_d \coloneqq \frac{d}{2}\log^2 \alpha_d + \sum_{r \geqslant 1} \frac{\alpha_d^{rd}}{r^2}.$$

Then,

$$\lim_{n\to\infty} \left(q_d^{(a)}(n) - Q_m^{(m_1,m_2)}(n)\right) = +\infty \text{ if } m > \left\lfloor \frac{\pi^2}{3A_d} \right\rfloor,$$
$$\lim_{n\to\infty} \left(q_d^{(a)}(n) - Q_m^{(m_1,m_2)}(n)\right) = -\infty \text{ if } m \leqslant \left\lfloor \frac{\pi^2}{3A_d} \right\rfloor.$$

This result considers the relationship between d-distinctness and equivalence modulo m when we allow d and m to vary independently of each other. In Section 4.4, we shall explore this in the fixed perimeter setting.

For S a subset of the positive integers, let $p^S(n)$ denote the number of partitions of n with no parts coming from S, S_0 denote the empty set, and S_k denote the set of integers $\{1, 2, \ldots, k\}$. Banerjee et al. [11, Theorems 1.5-1.7] extend the parity bias result (10) of Kim, Kim, and Lovejoy in the following.

Theorem 2.3 (Banerjee et al. [11], 2022). *For all $n > 7$, $n > 0$, and $n > 8$, respectively,*

$$p_o^{\{1\}}(n) < p_e^{\{1\}}(n), \tag{16}$$
$$p_o^{\{2\}}(n) > p_e^{\{2\}}(n), \tag{17}$$
$$p_o^{\{S_2\}}(n) > p_e^{\{S_2\}}(n). \tag{18}$$

We may also consider parity bias with a fixed degree of bias between even and odd parts. Let

$p(m, n) :-$ the number of partitions of n with exactly m more odd parts than even parts.

In 2023, Kim and Kim [22, Thm. 1] show that for all $n \notin \{2, 4, 5, 7, 9, 11, 13\}$,

$$p(1, n) > p(-1, n). \tag{19}$$

Let $d_o(n)$ (resp. $d_e(n)$) denote the number of partitions of n into distinct parts with more odd parts than even parts (resp. more even parts than odd parts). Originally conjectured by Kim, Kim and Lovejoy in [23], Banerjee et al. [11, Thm 1.4] prove the following parity bias result in the distinct setting.

Theorem 2.4 (Banerjee et al. [11], 2022). *For all $n > 19$,*

$$d_o(n) > d_e(n).$$

It is natural to try to extend parity bias results to a more general choice of modulus and residue classes. Let $1 \leq a < b \leq m$ and denote by $p_{a,b,m}(n)$ (resp. $p_{b,a,m}(n)$) the number of partitions of n with more parts congruent to $a \pmod m$ than parts congruent to $b \pmod m$ (resp. more parts congruent to $b \pmod m$ than parts congruent to $a \pmod m$). In 2022, Chern [13, Thm. 1.3] proved such a generalized bias result exists as follows.

Theorem 2.5 (Chern [13], 2022)**.** *For all $n \geq 1$ and $1 \leq a < b \leq m$,*

$$p_{a,b,m}(n) \geq p_{b,a,m}(n).$$

3. A Generalization of Franklin and Hovey: $O_{j,k,b}(n) = D_{j,k,b}(n)$

3.1. **Proofs of Theorem 1.1.** In this section, we provide two proofs of Theorem 1.1, one via generating functions and the other via a combinatorial bijection.

Generating function proof of Theorem 1.1. We first define

$$\mathscr{O}_{k,b}(z,q) \coloneqq \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} O_{j,k,b}(n) z^j q^n$$

and

$$\mathscr{D}_{k,b}(z,q) \coloneqq \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} D_{j,k,b}(n) z^j q^n.$$

We have

$$\mathscr{O}_{k,b}(z,q) = \prod_{n=1}^{\infty}\left(1 + zq^{kbn} + zq^{2(kbn)} + zq^{3(kbn)} + \cdots\right) \cdot \prod_{\substack{n=1\\ n\not\equiv 0\ (kb)}}^{\infty} \frac{1}{1-q^n}$$

$$= \prod_{n=1}^{\infty}\left(1 + \frac{zq^{kbn}}{1-q^{kbn}}\right)\prod_{n=1}^{\infty}\frac{1-q^{kbn}}{1-q^n}, \tag{20}$$

and

$$\begin{aligned}
&\mathscr{D}_{k,b}(z,q)\\
&= \prod_{n=1}^{\infty}\left(1 + q^{bn} + q^{2(bn)} + \cdots + q^{(k-1)(bn)} + zq^{k(bn)} + zq^{(k+1)(bn)} + \cdots\right) \cdot \prod_{\substack{n=1\\ n\not\equiv 0\ (b)}}^{\infty} \frac{1}{1-q^n}\\
&= \prod_{n=1}^{\infty}\left(1 + q^{bn} + q^{2(bn)} + \cdots + q^{(k-1)(bn)}\right)\left(1 + zq^{k(bn)} + zq^{2k(bn)} + \cdots\right)\prod_{n=1}^{\infty}\frac{1-q^{bn}}{1-q^n}\\
&= \prod_{n=1}^{\infty}\frac{1-q^{k(bn)}}{1-q^{bn}}\prod_{n=1}^{\infty}\left(1 + \frac{zq^{k(bn)}}{1-q^{k(bn)}}\right)\prod_{n=1}^{\infty}\frac{1-q^{bn}}{1-q^n}\\
&= \prod_{n=1}^{\infty}\frac{1-q^{kbn}}{1-q^n}\prod_{n=1}^{\infty}\left(1 + \frac{zq^{kbn}}{1-q^{kbn}}\right).
\end{aligned} \tag{21}$$

From (20) and (21), it is clear that $O_{j,k,b}(n) = D_{j,k,b}(n)$ as their generating functions are equal. □

Remark 3.1. *It is often beneficial to write $\mathscr{O}_{k,b}(z,q)$ and $\mathscr{D}_{k,b}(z,q)$ in the less simplified forms of (20) and (21). However, we may also write these generating functions succinctly using q-pochhammer notation as follows.*

$$\mathscr{O}_{k,b}(z,q) = \frac{((1-z)q^{kb};q^{kb})_\infty}{(q;q)_\infty} = \mathscr{D}_{k,b}(z,q),$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n).$$

We now provide a fully combinatorial proof of Theorem 1.1.

Bijective proof of Theorem 1.1. We define a family of bijections $\{\varphi_{k,b}\}_{bk=2}^{\infty}$ on $\mathcal{P}$, the set of all partitions. For $bk \geq 2$, consider a partition $\pi = (\pi_1, \pi_2, \ldots, \pi_r)$ counted by $O_{j,k,b}(n)$.

- Denote by $\overline{\pi}$ the partition composed of all parts $\equiv 0 (\text{mod } bk)$ in π. For each $\pi_i \in \overline{\pi}$, map π_i to k copies of parts of size $\frac{\pi_i}{k}$. Denote by $\overline{\eta}$ the partition composed of all parts converted in this way. Note that all parts in $\overline{\eta}$ appear at least k times and in fact appear a multiple of k times.
- Denote by $\widetilde{\pi}$ the partition composed of all parts appearing at least k time in π that are $\equiv 0 (\text{mod } b)$ but $\not\equiv 0 (\text{mod } bk)$. For each $\pi_i \in \widetilde{\pi}$, combine (add) k copies of the part π_i to create a new part of size $k\pi_i$. Repeat this process until no part appears more than $k-1$ times. Denote by $\widetilde{\eta}$ the partition composed of all parts converted in this way. Note that all parts in $\widetilde{\eta}$ are divisible by bk and appear less than k times.
- Denote by $\hat{\pi}$ the partition composed of all remaining parts in π. These parts are either not divisible by b or are divisible by b but appear less than k times. Map each part $\pi_i \in \hat{\pi}$ to itself. Denote by $\hat{\eta}$ the partition composed of these parts.

Finally, let $\varphi_{k,b}(\pi) \coloneqq \eta = \overline{\eta} \cup \widetilde{\eta} \cup \hat{\eta}$, where the union $\overline{\eta} \cup \widetilde{\eta} \cup \hat{\eta}$ consists of all parts in $\overline{\eta}$, $\widetilde{\eta}$, and $\hat{\eta}$ arranged in nonincreasing order. Via this map $\varphi_{k,b}$, the number of parts divisible by bk in π is equal to the number of parts divisible by b that appear at least k times in η.

The inverse map acts on a partition $\eta = (\eta_1, \eta_2, \ldots, \eta_r)$ as follows. Each distinct part η_d occurs $m_d = t_d k + l_d$ times, where $l_d < k$ and at most one of t_d, l_d are zero.

- Denote by $\overline{\eta}$ the partition composed of all $t_d k$ parts of those η_d for which $t_d > 0$. For each part $\eta_i \in \overline{\eta}$, combine k copies of the part η_i to create a new part of size $k\eta_i$. Repeat this process until no part appears more than $k-1$ times. The resulting partition is given exactly by $\overline{\pi}$.
- Denote by $\widetilde{\eta}$ the partition composed of all parts in η that are $\equiv 0 \ (\text{mod } bk)$. For each part $\eta_i \in \widetilde{\eta}$, map η_i to k copies of equal parts $\frac{\eta_i}{k}$. The resulting partition is given exactly by $\widetilde{\pi}$.
- Denote by $\hat{\eta}$ the partition composed of all remaining parts $\eta_i \notin \{\overline{\eta}, \widetilde{\eta}\}$. Map each $\eta_i \in \hat{\eta}$ to itself. The resulting partition is given exactly by $\hat{\pi}$.

Observe that $\varphi_{k,b}^{-1}(\eta) = \overline{\pi} \cup \widetilde{\pi} \cup \hat{\pi}$. Thus the function is indeed a bijection. □

We demonstrate below an example of the above bijection.

Example 3.2. We list the all the partitions counted by $O_{3,2,2}(29)$ and $D_{3,2,2}(29)$, respectively. Each row demonstrates the one-to-one correspondence via $\varphi_{2,2}$.

$\mathcal{O}_{3,2,2}(29)$	$\mathcal{D}_{3,2,2}(29)$
$(16, 8, 4, 1)$	$(8, 8, 4, 4, 2, 2, 1)$
$(12, 8, 5, 4)$	$(6, 6, 5, 4, 4, 2, 2)$
$(12, 8, 4, 4, 1)$	$(6, 6, 4, 4, 2, 2, 2, 2, 1)$
$(12, 8, 4, 3, 2)$	$(6, 6, 4, 4, 3, 2, 2, 2)$
$(12, 8, 4, 3, 1, 1)$	$(6, 6, 4, 4, 3, 2, 2, 1, 1)$
$(12, 8, 4, 2, 2, 1)$	$(6, 6, 4, 4, 4, 2, 2, 1)$
$(12, 8, 4, 2, 1, 1, 1)$	$(6, 6, 4, 4, 2, 2, 2, 1, 1, 1)$
$(12, 8, 4, 1, 1, 1, 1, 1)$	$(6, 6, 4, 4, 2, 2, 1, 1, 1, 1, 1)$

3.2. **A Beck-Type Companion Identity for Theorem 1.1.** In this section we prove Theorem 1.3. Recall from the introduction that

$$E(O_{j,k,b}(n), D_{j,k,b}(n)) = \sum_{\pi \in \mathcal{O}_{j,k,b}(n)} \ell(\pi) - \sum_{\pi \in \mathcal{D}_{j,k,b}(n)} \ell(\pi),$$

where $\ell(\pi)$ is the number of parts in the partition π.

We will first prove Theorem 1.3 in a more direct manner by adapting the generating functions for $O_{j,k,b}(n)$ and $D_{j,k,b}(n)$.

Proof of Theorem 1.3. Recall the generating functions for $O_{j,k,b}(n)$ and $D_{j,k,b}(n)$ given by (20) and (21). We have

$$\mathscr{O}_{k,b}(z, q) = \prod_{n=1}^{\infty} \left(1 + \frac{zq^{bkn}}{1 - q^{bkn}}\right) \prod_{n=1}^{\infty} \frac{1 - q^{bkn}}{1 - q^n},$$
$$\mathscr{D}_{k,b}(z, q) = \prod_{n=1}^{\infty} \left(1 + \frac{zq^{bkn}}{1 - q^{bkn}}\right) \prod_{n=1}^{\infty} \frac{1 - q^{bkn}}{1 - q^n}.$$

Now define

$$\mathscr{O}_{k,b}(z, w, q) := \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} O_{j,k,b}(n) z^j w^\ell q^n$$

and

$$\mathscr{D}_{k,b}(z, w, q) := \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} D_{j,k,b}(n) z^j w^\ell q^n.$$

We introduce the variable w to keep track of the total number of parts ℓ for each partition. Note that in $\mathscr{O}_{k,b}(z, q)$, the $1 - q^{kbn}$ term excludes terms divisible by kb, and in $\mathscr{D}_{k,b}(z, q)$, the same term requires parts divisible by b to appear $\geq k$ times. (This is why the w is raised to the power of k in $\mathscr{D}_{k,b}(z, w, q)$ and not $\mathscr{O}_{k,b}(z, w, q)$.) Adjusting the generating functions accordingly, we now have

$$\mathscr{O}_{k,b}(z,w,q) = \prod_{n=1}^{\infty}\left(1+\frac{zwq^{bkn}}{1-wq^{bkn}}\right)\prod_{n=1}^{\infty}\frac{1-wq^{bkn}}{1-wq^{n}} = \prod_{n=1}^{\infty}\left(\frac{1-wq^{bkn}+zwq^{bkn}}{1-wq^{n}}\right) \tag{22}$$

and

$$\mathscr{D}_{k,b}(z,w,q) = \prod_{n=1}^{\infty}\left(1+\frac{zw^kq^{bkn}}{1-w^kq^{bkn}}\right)\prod_{n=1}^{\infty}\frac{1-w^kq^{bkn}}{1-wq^{n}} = \prod_{n=1}^{\infty}\left(\frac{1-w^kq^{bkn}+zw^kq^{bkn}}{1-wq^{n}}\right). \tag{23}$$

Now, to determine $E(O_{j,k,b}(n), D_{j,k,b}(n))$, we first take the partial derivatives of (22) and (23) with respect to w and evaluate at $w=1$. This will weight each partition by its number of parts ℓ. After doing so, we take the difference to be left with the generating function for $E(O_{j,k,b}(n), D_{j,k,b}(n))$. We have

$$\begin{aligned}
&\left.\frac{\partial}{\partial w}\right|_{w=1}\mathscr{O}_{k,b}(z,w,q)\\
&\quad=\sum_{n=1}^{\infty}\left(\frac{(-q^{bkn}+zq^{bkn})(1-wq^n)-(1-wq^{bkn}+zwq^{bkn})(-q^n)}{(1-wq^n)^2}\right)\\
&\qquad\cdot\left.\prod_{\substack{m=1\\ m\neq n}}^{\infty}\left(\frac{1-wq^{kbm}+zwq^{kbm}}{1-wq^m}\right)\right|_{w=1}\\
&\quad=\sum_{n=1}^{\infty}\left(\frac{q^n-q^{bkn}+zq^{bkn}}{(1-q^n)(1-(1-z)q^{bkn})}\right)\prod_{m=1}^{\infty}\frac{1-(1-z)q^{bkm}}{1-q^m}\\
&\quad=\sum_{n=1}^{\infty}\left(\frac{q^n-(1-z)q^{bkn})}{(1-q^n)(1-(1-z)q^{bkn}}\right)\mathscr{O}_{k,b}(z,q),
\end{aligned}$$

$$\begin{aligned}
&\left.\frac{\partial}{\partial w}\right|_{w=1}\mathscr{D}_{k,b}(z,w,q)\\
&\quad=\sum_{n=1}^{\infty}\left(\frac{(-kw^{k-1}q^{bkn}+kzw^{k-1}q^{bkn})(1-wq^n)-(1-w^kq^{bkn}+zw^kq^{bkn})(-q^n)}{(1-wq^n)^2}\right)\\
&\qquad\cdot\left.\prod_{\substack{m=1\\ m\neq n}}^{\infty}\left(\frac{1-w^kq^{kbm}+zw^kq^{kbm}}{1-wq^m}\right)\right|_{w=1}\\
&\quad=\sum_{n=1}^{\infty}\left(\frac{q^n-k(q^{kbn}-zq^{kbn})+(k-1)(q^{(kb+1)n}-zq^{(kb+1)n})}{(1-q^n)(1-(1-z)q^{bkn})}\right)\prod_{m=1}^{\infty}\frac{1-(1-z)q^{bkm}}{1-q^m}\\
&\quad=\sum_{n=1}^{\infty}\left(\frac{q^n-k(1-z)q^{bkn}+(k-1)(1-z)q^{bkn}q^n}{(1-q^n)(1-(1-z)q^{bkn})}\right)\mathscr{O}_{k,b}(z,q).
\end{aligned}$$

After subtracting, we have

$$\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}E(O_{j,k,b}(n),D_{j,k,b}(n))z^jq^n$$

$$=\sum_{n=1}^{\infty}\left(\frac{(k-1)(1-z)q^{bkn}-(k-1)(1-z)q^{bkn}q^n}{(1-q^n)(1-(1-z)q^{bkn})}\right)\mathscr{O}_{k,b}(z,q)$$

$$=(k-1)(1-z)\sum_{n=1}^{\infty}\left(\frac{q^{bkn}(1-q^n)}{(1-q^n)(1-(1-z)q^{bkn})}\right)\prod_{m=1}^{\infty}\left(\frac{1-(1-z)q^{bkm}}{1-q^m}\right)$$

$$=(k-1)\left[(1-z)\sum_{n=1}^{\infty}\frac{q^{bkn}}{1-q^{bkn}}\prod_{\substack{m=1\\m\neq n}}^{\infty}\left(\frac{1-(1-z)q^{bkm}}{1-q^{bkm}}\right)\prod_{m=1}^{\infty}\frac{1-q^{bkm}}{1-q^m}\right] \tag{24}$$

$$=(k-1)\left[(1-z)\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}(j+1)O_{j+1,k,b}(n)z^jq^n\right] \tag{25}$$

$$=(k-1)\left[\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}((j+1)O_{j+1,k,b}(n)-(j)O_{j,k,b}(n))z^jq^n\right].$$

The second to last equality (25) comes in the same manner as in Ballantine & Welch [8, pf. of Thm. 4]. The $\frac{q^{bkn}}{1-q^{bkn}}$ in (24) corresponds to the part(s) of size bkn for a partition in $\mathcal{O}_{j+1,k,b}(n)$. The following product corresponds to the other j parts which are divisible by bk along with the weight z^j. For a partition in $O_{j+1,k,b}(n)$, there are $j+1$ parts which can be contributed by the first term, implying that π is counted $j+1$ times in the coefficient of z^jq^n. Thus,

$$\frac{1}{k-1}E(O_{j,k,b}(n),D_{j,k,b}(n))=(j+1)O_{j+1,k,b}(n)-jO_{j,k,b}(n)=(j+1)D_{j+1,k,b}(n)-jD_{j,k,b}(n).$$

□

We now present perhaps a more illuminating approach to prove Theorem 1.3. We use the following definitions to adapt the modular refinement used by Ballantine and Welch in [8]. Let $k\geq 2$ and $0\leq t\leq k-1$. Given a partition π, let

$$\ell_t(\pi)=\text{ the number of parts }\equiv bt(\text{mod }bk)\text{ in }\pi.$$

Now, for a part $i\in\pi$ that appears s_i times, we shall refer to s_i (mod k) as the *residual multiplicity* of i in π, denoted by

$$r_\pi(i). \tag{26}$$

Similarly, we refer to $\lfloor\frac{s_i}{k}\rfloor$ as the *nonresidual multiplicity* of i in π, denoted

$$\tilde{r}_\pi(i). \tag{27}$$

Note that $s_i = k\tilde{r}_\pi(i) + r_\pi(i)$. We let $\bar{\ell}_t(\pi)$ count the number of different parts $\equiv 0 \pmod{b}$ in π with $r_\pi(i) \geq t$. Note that these are the same counting functions as in [8], however, we are only considering the parts that are divisible by b. This is due to the nature of adapting $O_{j,k}(n)$ and $D_{j,k}(n)$ to $O_{j,k,b}(n)$ and $D_{j,k,b}(n)$. We now define a refinement of $E(O_{j,k,b}(n), D_{j,k,b}(n))$. For a given $1 \leq t \leq k-1$, let

$$E_{j,k,b,t}(n) \coloneqq \sum_{\pi \in \mathcal{O}_{j,k,b}(n)} (\ell_t(\pi) - \ell_0(\pi)) - \sum_{\pi \in \mathcal{D}_{j,k,b}(n)} \bar{\ell}_t(\pi).$$

To help construct our second proof of Theorem 1.3, we introduce the following lemma.

Lemma 3.3. *For integers n, j, k, b, t with $n, j \geq 0$, $k \geq 2$, $b \geq 1$ and $1 \leq t \leq k-1$, we have*

$$\begin{aligned} E_{j,k,b,t}(n) &= (j+1)O_{j+1,k,b}(n) - jO_{j,k,b}(n) \\ &= (j+1)D_{j+1,k,b}(n) - jD_{j,k,b}(n). \end{aligned}$$

Proof. We fix $k \geq 2$ and $1 \leq t \leq k-1$. We shall denote by $\mathcal{O}_{j,k,b,t}(m,n)$ the subset of partitions in $\mathcal{O}_{j,k,b}(n)$ in which exactly m parts are equivalent to $bt (\bmod bk)$. We also denote by $\mathcal{D}_{j,k,b,t}(m,n)$ the subset of partitions in $\mathcal{D}_{j,k,b}(n)$ with m parts with residual multiplicity $i \equiv 0 (\bmod b)$ that also satisfy $r_\pi(i) \geq t$. Similarly let $\mathcal{O}_{j,k,b,0}(m,n)$ be the subset of partitions in $\mathcal{O}_{j,k,b}$ with m parts divisible by bk.

As has been standard thus far we allow

$$O_{j,k,b,t}(m,n) = |\mathcal{O}_{j,k,b,t}(m,n)|$$

and

$$D_{j,k,b,t}(m,n) = |\mathcal{D}_{j,k,b,t}(m,n)|.$$

We define the following trivariate generating functions

$$\begin{aligned} \mathscr{O}_{k,b,t}(z,w,q) &\coloneqq \sum_{n=0}^\infty \sum_{j=0}^\infty \sum_{m=0}^\infty O_{j,k,b,t}(m,n) z^m w^j q^n, \\ \mathscr{D}_{k,b,t}(z,w,q) &\coloneqq \sum_{n=0}^\infty \sum_{j=0}^\infty \sum_{m=0}^\infty D_{j,k,b,t}(m,n) z^m w^j q^n, \\ \mathscr{O}_{k,b,0}(z,w,q) &\coloneqq \sum_{n=0}^\infty \sum_{j=0}^\infty \sum_{m=0}^\infty O_{j,k,b,0}(m,n) z^m w^j q^n. \end{aligned}$$

As the variable w is being used to count the number of parts, it is evident by our definitions that

$$\sum_{n=0}^\infty \sum_{j=0}^\infty E_{j,k,b,t}(n) z^j q^n = \frac{\partial}{\partial w}\bigg|_{w=1} (\mathscr{O}_{k,b,t}(z,w,q) - \mathscr{O}_{k,b,0}(z,w,q) - \mathscr{D}_{k,b,t}(z,w,q)).$$

We shall derive each generating function $\mathscr{O}_{k,b,t}(z,w,q)$, $\mathscr{O}_{k,b,0}(z,w,q)$, and $\mathscr{D}_{k,b,t}(z,w,q)$, take each of their partial derivatives at $w=1$, then subtract them to determine $E_{j,k,b,t}(n)$.

In $\mathscr{O}_{k,b,t}(z,w,q)$ we include a w for terms that are equivalent to bt modulo bk. In $\mathscr{D}_{k,b,t}(z,w,q)$ we include a w for terms divisible by b whose residual multiplicity is $\geq t$.

In $\mathscr{O}_{k,b,0}(z,w,q)$ we include a w for terms divisible by bk. We have,

$$\begin{aligned}
&\mathscr{O}_{k,b,t}(z,w,q)\\
&\quad=\prod_{n=1}^{\infty}(1+zq^{bkn}+zq^{2bkn}+\cdots)\\
&\qquad\cdot\prod_{n=0}^{\infty}\frac{1}{(1-q^{bkn+1})\cdots(1-q^{bkn+bt-1})(1-wq^{bkn+bt})(1-q^{bkn+bt+1})\cdots(1-q^{bkn+bk-1})}\\
&\quad=\prod_{n=1}^{\infty}\left(1+\frac{zq^{kbn}}{1-q^{kbn}}\right)\prod_{n=1}^{\infty}\frac{1-q^{bkn}}{1-q^{n}}\prod_{n=0}^{\infty}\frac{1-q^{bkn+bt}}{1-wq^{bkn+bt}},\\
&\mathscr{D}_{k,b,t}(z,w,q)\\
&\quad=\prod_{n=1}^{\infty}\left(1+\frac{zq^{bkn}}{1-q^{bkn}}\right)\prod_{\substack{n=1\\ n\not\equiv 0\ (b)}}^{\infty}\frac{1}{1-q^{n}}\\
&\qquad\cdot\prod_{n=1}^{\infty}(1+q^{bn}+\cdots+q^{(t-1)bn}+wq^{tbn}+wq^{(t+1)bn}+\cdots+wq^{(k-1)bn})\\
&\quad=\prod_{n=1}^{\infty}\left(1+\frac{zq^{bkn}}{1-q^{bkn}}\right)\prod_{n=1}^{\infty}\frac{1-q^{bn}}{1-q^{n}}\prod_{n=1}^{\infty}\frac{1-q^{btn}+wq^{btn}-wq^{bkn}}{1-q^{bn}},
\end{aligned}$$

and

$$\mathscr{O}_{k,b,0}(z,w,q)=\prod_{n=1}^{\infty}\left(1+\frac{wzq^{bkn}}{1-wq^{bkn}}\right)\prod_{\substack{n=1\\ n\not\equiv 0\ (kb)}}^{\infty}\frac{1}{1-q^{n}}=\prod_{n=1}^{\infty}\left(\frac{1-(1-z)wq^{bkn}}{1-wq^{bkn}}\right)\prod_{n=1}^{\infty}\frac{1-q^{bkn}}{1-q^{n}}.$$

Now, differentiation gives

$$\begin{aligned}
&\frac{\partial}{\partial w}\bigg|_{w=1}\mathscr{O}_{k,b,t}(z,w,q)\\
&\quad=\prod_{n=1}^{\infty}\left(1+\frac{zq^{bkn}}{1-q^{bkn}}\right)\prod_{n=1}^{\infty}\frac{1-q^{bkn}}{1-q^{n}}\sum_{m=0}^{\infty}\left(\frac{q^{bkm+bt}(1-q^{bkm+bt})}{(1-q^{bkm+bt})^2}\prod_{\substack{n=0\\ n\neq m}}^{\infty}\frac{1-q^{bkn+bt}}{1-q^{bkn+bt}}\right)\\
&\quad=\prod_{n=1}^{\infty}\left(1+\frac{zq^{bkn}}{1-q^{bkn}}\right)\prod_{n=1}^{\infty}\frac{1-q^{bkn}}{1-q^{n}}\sum_{m=0}^{\infty}\frac{q^{bkm+bt}}{1-q^{bkm+bt}}\\
&\quad=\prod_{n=1}^{\infty}\left(1+\frac{zq^{bkn}}{1-q^{bkn}}\right)\prod_{n=1}^{\infty}\frac{1-q^{bkn}}{1-q^{n}}\sum_{m=1}^{\infty}\frac{q^{btm}}{1-q^{bkm}}, \qquad (28)\\
&\frac{\partial}{\partial w}\bigg|_{w=1}\mathscr{D}_{k,b,t}(z,w,q)
\end{aligned}$$

$$=\prod_{n=1}^{\infty}\left(1+\frac{zq^{bkn}}{1-q^{bkn}}\right)\prod_{n=1}^{\infty}\frac{1-q^{bn}}{1-q^{n}}\sum_{m=1}^{\infty}\left(\frac{q^{btm}-q^{bkm}}{1-q^{bm}}\prod_{\substack{n=1\\n\neq m}}^{\infty}\frac{1-q^{btn}+q^{btn}-q^{bkn}}{1-q^{bn}}\right)$$

$$=\prod_{n=1}^{\infty}\left(1+\frac{zq^{bkn}}{1-q^{bkn}}\right)\prod_{n=1}^{\infty}\frac{1-q^{bkn}}{1-q^{n}}\sum_{m=1}^{\infty}\frac{q^{btm}-q^{bkm}}{1-q^{bkm}}, \tag{29}$$

$$\frac{\partial}{\partial w}\bigg|_{w=1}\mathscr{O}_{k,b,0}(z,w,q)=\prod_{n=1}^{\infty}\frac{1-q^{bkn}}{1-q^{n}}$$

$$\cdot\sum_{m=1}^{\infty}\left(\frac{-(1-z)q^{bkm}(1-q^{bkm})+q^{bkm}(1-(1-z)q^{bkm})}{(1-q^{bkm})^{2}}\prod_{\substack{n=1\\n\neq m}}^{\infty}\frac{1-(1-z)q^{bkn}}{1-q^{bkn}}\right)$$

$$=\prod_{n=1}^{\infty}\left(1+\frac{zq^{bkn}}{1-q^{bkn}}\right)\prod_{n=1}^{\infty}\frac{1-q^{bkn}}{1-q^{n}}\sum_{m=1}^{\infty}\frac{q^{bkm}-(1-z)q^{bkm}}{(1-q^{bkm})(1-(1-z)q^{bkm})}. \tag{30}$$

Note that our equality in (28) comes as in Ballantine & Welch [8, pf. of Thm. 4] from the fact that

$$\sum_{n=0}^{\infty}\frac{q^{bkn+bt}}{1-q^{bkn+bt}}=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}q^{m(bkn+bt)}=\sum_{m=1}^{\infty}q^{btm}\sum_{n=0}^{\infty}q^{nbkm}=\sum_{m=1}^{\infty}\frac{q^{btm}}{1-q^{brm}}.$$

We now subtract (29) and (30) from (28) to obtain

$$\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}E_{j,k,b,t}(n)z^{j}q^{n}=\prod_{n=1}^{\infty}\left(1+\frac{zq^{bkn}}{1-q^{bkn}}\right)\prod_{n=1}^{\infty}\frac{1-q^{bkn}}{1-q^{n}}$$

$$\cdot\sum_{m=1}^{\infty}\left(\frac{q^{btm}}{1-q^{bkm}}-\frac{q^{btm}-q^{bkm}}{1-q^{bkm}}-\frac{q^{bkm}-(1-z)q^{bkm}}{(1-q^{bkm})(1-(1-z)q^{bkm})}\right). \tag{31}$$

The summand in (31) simplifies as

$$\frac{q^{btm}}{1-q^{bkm}}-\frac{q^{btm}-q^{bkm}}{1-q^{bkm}}-\frac{q^{bkm}-(1-z)q^{bkm}}{(1-q^{bkm})(1-(1-z)q^{bkm})}=\frac{(1-z)q^{bkm}}{(1-(1-z)q^{bkm})}.$$

Thus,

$$\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}E_{j,k,b,t}(n)z^{j}q^{n}=\prod_{n=1}^{\infty}\left(1+\frac{zq^{bkn}}{1-q^{bkn}}\right)\prod_{n=1}^{\infty}\frac{1-q^{bkn}}{1-q^{n}}\sum_{m=0}^{\infty}\frac{(1-z)q^{bkm}}{(1-(1-z)q^{bkm})}$$

$$=(1-z)\sum_{m=1}^{\infty}\left(\frac{q^{bkm}}{1-q^{bkm}}\prod_{\substack{n=1\\n\neq m}}^{\infty}\left(1+\frac{zq^{bkn}}{1-q^{bkn}}\right)\prod_{n=1}^{\infty}\frac{1-q^{bkn}}{1-q^{n}}\right)$$

$$=(1-z)\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}(j+1)O_{j+1,k,b}(n)z^{j}q^{n}$$

$$=\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\left((j+1)O_{j+1,k,b}(n)-jO_{j,k,b}(n)\right)z^jq^n.$$

Here, the second to last equality follows as in the previous proof of Theorem 1.3 as well as in [8, pf of Thm. 4]. So we have our result,

$$E_{j,k,b,t}(n)=(j+1)O_{j+1,k,b}(n)-jO_{j,k,b}(n)=(j+1)D_{j+1,k,b}(n)-jD_{j,k,b}(n).$$

□

We now provide an analytic proof of Theorem 1.3 through the use of Lemma 3.3.

Proof of Theorem 1.3 through a modular refinement. For $k\geq 2$, we will sum over $1\leq t\leq k-1$ in Lemma 3.3. We have that

$$\begin{aligned}\sum_{t=1}^{k-1}E_{j,k,b,t}(n)=&(r-1)((j+1)O_{j+1,k,b}(n)-jO_{j,k,b}(n))\\ =&(r-1)((j+1)D_{j+1,k,b}(n)-jD_{j,k,b}(n)).\end{aligned}$$

It remains to be shown, however, that

$$E(O_{j,k,b}(n),D_{j,k,b}(n))=\sum_{t=1}^{k-1}E_{j,k,b,t}(n).$$

We claim that we need only count the number of parts $\equiv 0 \pmod b$ when considering $E(O_{j,k,b}(n),D_{j,k,b}(n))$, as the number of parts $\not\equiv 0 \pmod b$ in $\mathcal{O}_{j,k,b}(n)$ and $\mathcal{D}_{j,k,b}(n)$ are equinumerous. To see this, consider the following generating function, in which we let w count the number of parts $\not\equiv 0 \pmod b$. For both $O_{j,k,b}(n)$ and $D_{j,k,b}(n)$, this takes the form

$$\prod_{n=1}^{\infty}\left(1+\frac{zq^{bkn}}{1-q^{bkn}}\right)\prod_{n=1}^{\infty}\frac{1-q^{bkn}}{1-q^{bn}}\prod_{n=1}^{\infty}\frac{1-wq^{bn}}{1-wq^{n}}.$$

Thus, we shall only consider the parts divisible by b. We have that

$$\begin{aligned}\sum_{t=1}^{k-1}E_{j,k,b,t}(n)&=\sum_{t=1}^{k-1}\left(\sum_{\pi\in\mathcal{O}_{j,k,b}(n)}(\ell_t(\pi)-\ell_0(\pi))-\sum_{\pi\in\mathcal{D}_{j,k,b}(n)}\bar{\ell}_t(\pi)\right)\\ &=\sum_{\pi\in\mathcal{O}_{j,k,b}(n)}\ell^b(\pi)-k\sum_{\pi\in\mathcal{O}_{j,k,b}(n)}\ell_0(\pi)-\sum_{t=1}^{k-1}\sum_{\pi\in\mathcal{D}_{j,k,b}(n)}\bar{\ell}_t(\pi),\end{aligned}$$

where $\ell^b(\pi)$ is the number of parts $\equiv 0 \pmod b$ in a partition π. For a given partition $\pi\in\mathcal{D}_{j,k,b}(n)$, each unique part $i\in\pi$ is counted in

$$\sum_{t=1}^{k-1}\sum_{\pi\in\mathcal{D}_{j,k,b}(n)}\bar{\ell}_t(\pi)$$

as many times as its residual multiplicity $r_\pi(i)$, as defined in (26). Alternatively,

$$\sum_{t=1}^{k-1} \sum_{\pi \in \mathcal{D}_{j,k,b}(n)} \bar{\ell}_t(\pi) = \sum_{\pi \in \mathcal{D}_{j,k,b}(n)} \sum_{\substack{i \in \pi \\ b|i}} r_\pi(i).$$

If s_i is the multiplicity of a part i in the partition π, we have that $s_i = k\tilde{r}_\pi(i) + r_\pi(i)$, where $\tilde{r}_\pi(i)$ is the nonresidual multiplicity (27) of i in π. It is evident that

$$\begin{aligned} \sum_{\pi \in \mathcal{D}_{j,k,b}(n)} \ell^b(\pi) &= \sum_{\pi \in \mathcal{D}_{j,k,b}(n)} \sum_{\substack{i \in \pi \\ b|i}} (k\tilde{r}_\pi(i) + r_\pi(i)) \\ &= k \sum_{\pi \in \mathcal{D}_{j,k,b}(n)} \sum_{\substack{i \in \pi \\ b|i}} \tilde{r}_\pi(i) + \sum_{\pi \in \mathcal{D}_{j,k,b}(n)} \sum_{\substack{i \in \pi \\ b|i}} r_\pi(i). \end{aligned}$$

It therefore suffices to show

$$\sum_{\pi \in \mathcal{O}_{j,k,b}(n)} \ell_0(\pi) = \sum_{\pi \in \mathcal{D}_{j,k,b}(n)} \sum_{\substack{i \in \pi \\ b|i}} \tilde{r}_\pi(i).$$

We shall denote by $\overline{\mathcal{D}}_{k,b}(m,n)$ the subset of partitions $\pi \in \mathcal{D}_{k,b}(n)$ in which the sum of $\tilde{r}_\pi(i)$ over unique parts $i \equiv 0 \pmod b$ in π is equal to m, where $\overline{D}_{j,k,b}(m,n) = |\overline{\mathcal{D}}_{j,k,b}(m,n)|$. Let

$$\overline{\mathscr{D}}_{k,b}(z,w,q) := \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \overline{D}_{j,k,b}(m,n) z^j w^m q^n.$$

We then have

$$\overline{\mathscr{D}}_{k,b}(z,w,q) = \prod_{n=1}^{\infty} \left(1 + \frac{wzq^{bkn}}{1 - wq^{bkn}}\right) \prod_{n=1}^{\infty} \frac{1 - q^{bkn}}{1 - q^n} = \mathscr{O}_{k,b,0}(z,w,q).$$

It is now evident that

$$\sum_{\pi \in \mathcal{D}_{j,k,b}(n)} \sum_{\substack{i \in \pi \\ b|i}} \tilde{r}_\pi(i) = \frac{\partial}{\partial w}\bigg|_{w=1} \overline{\mathscr{D}}_{k,b}(z,w,q) = \frac{\partial}{\partial w}\bigg|_{w=1} \mathscr{O}_{k,b,0}(z,w,q) = \sum_{\pi \in \mathcal{O}_{j,k,b}(n)} \ell_0(\pi).$$

Thus,

$$E(O_{j,k,b}(n), D_{j,k,b}(n)) = \sum_{t=1}^{k-1} E_{j,k,b,t}(n).$$

□

4. Fixed Perimeter Analogues of Regular Partition Results

We now shift focus to the fixed perimeter partition setting. Recall, as outlined in the introduction, the perimeter of a partition is given by the number of dots on the north and west edges of its corresponding diagram. We often consider the perimeter by the arm length $\alpha(\pi)$ plus the leg length $\lambda(\pi)$ minus one, as the northwestern most dot is counted by both α and λ.

Within this section, it is often useful to consider a partition by its *profile* rather than by the constituting parts. Each partition is in bijection with a unique profile, which corresponds to the set of southmost and eastmost edges. From this profile, we shall construct a *word* for a given partition π, say w_π, that consists of $\{E, N\}$. Beginning at the southwest corner of the partition, we trace along the profile. Each horizontal edge contributes an E to our word, while each vertical edge contributes an N to our word. Note that each word corresponding to a partition with perimeter n is of length $n+1$ and must always begin with E and end with N.

Within a word w_π, the number of Es is precisely the largest part of π (arm length) $\alpha(\pi)$. Similarly, the number of Ns is precisely the number of parts of π (leg length) $\lambda(\pi)$. This gives further intuition into the length of our word for a given partition, as $\alpha + \lambda = n + 1$.

4.1. Analogue of a Special Case in Franklin's Identity.

We present two proofs of Theorem 1.4–the first a bijective proof generalizing Fu and Tang's bijection [17] and the second a generating function proof.

Recall that we have defined

$$FO_{j,k}(n) \coloneqq r(n \mid \text{exactly } j \text{ parts are divisible by } k),$$
$$FD_{j,k}(n) \coloneqq r(n \mid \text{exactly } j \text{ parts appear at least } k \text{ times}).$$

We also allow $\mathcal{FO}_{j,k}(n)$ (resp. $\mathcal{FD}_{j,k}(n)$) to be the set of partitions counted by $FO_{j,k}(n)$ (resp. $FD_{j,k}(n)$).

Bijective Proof of Theorem 1.4. We shall first recall the bijection presented by Fu and Tang [17], using the language of words referring to the profile of a partition. Given a partition $\pi \in \mathcal{FD}_{0,2}(n)$ we shall apply a function $\gamma : \mathcal{FD}_{0,2}(n) \to \mathcal{FO}_{0,2}(n)$ in which $\gamma(\pi)$ is defined by the following mapping

- Initial $E \mapsto E$
- $NE \mapsto EE$
- Any E which is preceded by an $E \mapsto N$
- Final $N \mapsto N$

The authors prove that $\gamma(\pi) \in \mathcal{FO}_{0,2}(n)$, as any E after the initial move must occur in a pair, ensuring that all parts are odd. Note that this is a bijection, with the inverse γ^{-1} defined as follows.

- Initial $E \mapsto E$
- $EE \mapsto NE$
- All N but final $\mapsto E$
- Final $N \mapsto N$

It is evident that $\gamma^{-1}(\gamma(\pi)) = \pi$, proving the bijection. We adapt this to obtain a bijection

$$\xi : \mathcal{FD}_{j,2}(n) \to \mathcal{FO}_{j,2}(n)$$

allowing for $j \geq 0$ non-distinct (resp. even) parts.

For $\pi \in \mathcal{FD}_{j,2}(n)$, repeated parts are identifiable by a tuple of $i \geq 2$ consecutive Ns. Likewise, for $\pi \in \mathcal{FD}_{j,2}(n)$, even parts are distinguished by an odd number of Es followed by $i \geq 1$ consecutive Ns, omitting the initial move E in our count. In adapting γ to create ξ, we can extend the rule mapping NE to EE to include these possibilities. For $\pi \in \mathcal{FD}_{j,2}(n)$, $\xi(\pi)$ is defined by the following mapping.

- First $E \mapsto E$
- Any $i+1$-tuple of $(i)N\ E \mapsto$ an $i+1$-tuple $E\ (i-1)N\ E$, for $i \geq 1$
- Any E which is preceded by an $E \mapsto N$
- Final t Ns $\mapsto \begin{cases} N & \text{if } t = 1 \\ E\ (t-1)N & \text{if } t > 1 \end{cases}$

Here, a part that occurs multiple times is converted into an even part, allowing for repetitions. We also consider the case when the largest part is repeated, converting it to an even part. Note that we allow $i = 1$, containing the mapping of $NE \to EE$. We define the inverse ξ^{-1} as follows.

- First $E \mapsto E$
- Any $i+1$-tuple of $E\ (i)N$ in which total preceding Es is odd $\mapsto$ an $i+1$-tuple $(i+1)N$, for $i \geq 0$
- Final $N \mapsto N$, if not considered in previous step
- Any remaining $N \mapsto E$

It is evident that $\xi^{-1}(\xi(\pi)) = \pi$, confirming the bijection between $\mathcal{FD}_{j,2}(n)$ and $\mathcal{FO}_{j,2}(n)$. Thus,

$$FO_{j,2}(n) = |\mathcal{FO}_{j,2}(n)| = |\mathcal{FD}_{j,2}(n)| = FD_{j,2}(n).$$

□

We now present a generating function proof of the same equality.

Generating function proof of Theorem 1.4. We first define the bivariate generating functions

$$\mathscr{FO}_k(z,q) = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} FO_{j,k}(n) z^j q^n,$$

$$\mathscr{FD}_k(z,q) = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} FD_{j,k}(n) z^j q^n.$$

Further, we introduce refinements based on largest part (arm length) α and number of parts (leg length) λ for a partition of perimeter n. We have

$$\begin{aligned} FO_{j,k}(\alpha,\lambda) &= FO_{j,k}(\alpha,\lambda,n) \\ &= r(n \mid \text{ j parts are} \equiv 0 (\text{mod } k) \text{ with largest part } \alpha \text{ and } \lambda \text{ parts}), \\ FD_{j,k}(\alpha,\lambda) &= FD_{j,k}(\alpha,\lambda,n) \end{aligned}$$

$$= r(n \mid \text{ j parts occur} \geq k \text{ times with largest part } \alpha \text{ and } \lambda \text{ parts}),$$

with quadvariate generating functions

$$\mathscr{FO}_k(x,y,z,q) \coloneqq \sum_{j=0}^{\infty}\sum_{\lambda=0}^{\infty}\sum_{\alpha=0}^{\infty} FO_{j,k}(\alpha,\lambda)x^{\alpha}y^{\lambda}z^{j}q^{\alpha+\lambda-1},$$
$$\mathscr{FD}_k(x,y,z,q) \coloneqq \sum_{j=0}^{\infty}\sum_{\lambda=0}^{\infty}\sum_{\alpha=0}^{\infty} FD_{j,k}(\alpha,\lambda)x^{\alpha}y^{\lambda}z^{j}q^{\alpha+\lambda-1}.$$

We fix $k=2$ for this proof but note that the above generating function definitions allow for generality. For $k=2$ we will show that

$$\mathscr{FO}_2(x,y,z,q) = \frac{xyq(1-yq+xzq)}{1-2yq-(x^2-y^2)q^2+(1-z)x^2yq^3}, \tag{32}$$

$$\mathscr{FD}_2(x,y,z,q) = \frac{xyq(1-(1-z)yq)}{1-(x+y)q+(1-z)xy^2q^3}. \tag{33}$$

Before we prove (32) and (33), we first note that taking $x=y=1$ in (32), (33) gives

$$\mathscr{FO}_2(1,1,z,q) = \mathscr{FO}_2(z,q) = \frac{q(1-(1-z)q)}{1-2q+(1-z)q^3} = \mathscr{FD}_2(z,q) = \mathscr{FD}_2(1,1,z,q). \tag{34}$$

Thus, for any $j,n \in \mathbb{N}$ it follows from (34) that

$$FO_{j,2}(n) = FD_{j,2}(n).$$

We now derive the generating functions in (32), (33), beginning with $\mathscr{FO}_2(x,y,z,q)$. We shall borrow a technique from Fu and Tang [17] when constructing fixed perimeter generating functions. In our generating function, the x, which is counting the largest part α, corresponds to any E in our word. Similarly the y, which is counting the number of parts λ, corresponds to any N in our word. We shall also allow z to track moves counted by j, as is done in the regular partition setting. In constructing our generating function, we shall consider a traversal of the profile of a partition counted by $FO_{j,k}(n)$, permitting valid choices of E, N.

Partitions in $\mathcal{FO}_{j,2}(n)$ (partitions with perimeter n containing j parts even) begin with a single move E followed by $i \geq 0$ moves N, which we will call the first *block* of the generating function. We include the final move N ending the profile of our partition in this block as well, as this move is likewise always present. Then this first block is generated by

$$xyq(1+yq+(yq)^2+\cdots) = \frac{xyq}{1-yq}.$$

We continue tracing the profile by single moves E followed by $i \geq 0$ copies of N. While tracing the profile, if the current total number of E's is even and is followed by any number of Ns, we shall increment the power of z by 1. Having an even number of total E's corresponds to including a part of even size for every move N. Thus, if we include any of these parts, we must increase the power of z, as we have included an even sized part.

So, including our step E prior, the block generating an even part takes the following form.

$$xq(1+zyq+z(yq)^2+\cdots) = xq\left(1+\frac{zyq}{1-yq}\right).$$

If the current total number of E's is odd, we allow it to move N unrestricted. This is generated by the term

$$xq(1+yq+(yq)^2+\cdots)=\frac{xq}{1-yq}.$$

We must also account for the possibility of having an even largest part. It is clear that the largest part must occur at least once, so the power of z must be incremented by 1. We thus require in our generating function that a largest even part is generated by the term

$$xq\frac{z}{1-yq}.$$

Now, we must put together these blocks of the generating function which we have derived. Let ω represent the initial block, a represent the generating function for an odd part, and b represent the generating function for an even part. The initial block includes an odd part, so following that, we must then alternate between even and odd parts, as we take moves E. Using our representatives, this will take the format

$$\omega(1+b+ab+ab^2+a^2b^2+a^2b^3+\cdots).$$

We shall now let c represent ending on an even part. Going off of our format, this equates to replacing a given b with c the first time it appears, as the most recently included part size in the profile is the largest part. After this replacement we have

$$\omega(1+c+ab+abc+a^2b^2+a^2b^2c+\dots)=\omega\frac{1+c}{1-ab}=\omega(1+c)(1-ab)^{-1}.$$

Substituting for our previously derived blocks, we have

$$\begin{aligned}\mathscr{F}\mathscr{O}_{j,2}(x,y,z,q)=&\frac{xyq}{1-yq}\left(1+\frac{xqz}{1-yq}\right)\left(1-\frac{xq}{1-yq}\cdot\frac{xq(1-(1-z)yq)}{1-yq}\right)^{-1}\\=&\frac{xyq(1-yq+xzq)}{(1-yq)^2-x^2q^2(1-(1-z)yq)}\\=&\frac{xyq(1-yq+xzq)}{1-2yq-(x^2-y^2)q^2+(1-z)x^2yq^3}.\end{aligned}$$

We shall now derive our generating function for $\mathscr{F}\mathscr{D}_2(x,y,z,q)$ using the same method to trace profiles for partitions in $\mathcal{FD}_{j,2}(n)$. We shall begin, as in the previous derivation, with a single move E followed by $i\geq 0$ moves N. If there are ≥ 2 steps N we must increment the power of z by 1, as this equates to repeating a part. This first block will be generated by

$$xq(1+yq+z(yq)^2+z(yq)^3+\cdots)=\frac{xq(1-(1-z)y^2q^2)}{1-yq}. \tag{35}$$

Unlike our previous generating function, we do not include a single step N for the completion of the partition. This is because we must be able to precisely determine the number of repetitions for all parts of our partition, including, of course, the largest part, and by simply taking a move N, this may cause an uncounted repetition. Before considering this, we will fill out the *body* of our partition, the movements between the first block and final block. Tracing through the profile, after any step E, we allow for $i\geq 0$ moves N, while incrementing the power of z if $i \geq 2$. The block generating this step is exactly the same as our first block, taking the form

$$\frac{xq(1-(1-z)y^2q^2)}{1-yq}.$$

We shall allow for i of these blocks in our generating function, for $i \geq 0$. So, the block generating the body of our partition will take the form

$$1+\frac{xq(1-(1-z)y^2q^2)}{1-yq}+\left(\frac{xq(1-(1-z)y^2q^2)}{1-yq}\right)^2+\cdots = \left(1-\frac{xq(1-(1-z)y^2q^2)}{1-yq}\right)^{-1}. \tag{36}$$

To ensure our final power of z is accurate, we will allow for the final block to include all repetitions of the largest part. To ensure that these have not been counted elsewhere, we first take a step E and require at least one step N, as our largest part must be present at least once. This is generated by

$$xyq(1+z(yq)+z(yq)^2+\cdots) = \frac{xyq(1-(1-z)yq)}{1-yq}. \tag{37}$$

We emit a q from this block to fix $n = \alpha + \lambda - 1$. We now put these blocks together; the initial block given by (35), the body block given by (36), and the final block given by (37). We have

$$\left(\frac{xq(1-(1-z)y^2q^2)}{1-yq}\right)\cdot\left(1-\frac{xq(1-(1-z)y^2q^2)}{1-yq}\right)^{-1}\cdot\left(\frac{xyq(1-(1-z)yq)}{1-yq}\right).$$

Note, however, that this generating function does not allow for partitions in which all parts are equal to 1, as the initial and final blocks require 2 steps E total. To remedy this, we allow for the final block to occur on its own, ensuring that z is still accurately incremented if the largest part is repeated. We have our final generating function.

$$\begin{aligned}
\mathscr{F}\mathscr{D}_2(x,y,z,q) =& \frac{xyq(1-(1-z)yq)}{1-yq} \\
&\cdot\left(1+\left(\frac{xq(1-(1-z)y^2q^2)}{1-yq}\right)\left(1-\frac{xq(1-(1-z)y^2q^2)}{1-yq}\right)^{-1}\right) \\
=& \frac{xyq(1-(1-z)yq)}{1-yq}\left(1-\frac{xq(1-(1-z)y^2q^2)}{1-yq}\right)^{-1} \\
=& \frac{xyq(1-(1-z)yq)}{1-yq-xq(1-(1-z)y^2q^2} \\
=& \frac{xyq(1-(1-z)yq)}{1-(x+y)q+(1-z)xy^2q^3}.
\end{aligned}$$

□

We note that the same technique can be used to derive generating functions for $FO_{j,k}(n)$ and $FD_{j,k}(n)$ for $k \geq 2$. However, for $k \geq 2$, the equality of $FO_{j,k}(n)$ and $FD_{j,k}(n)$ no longer holds, and the generating functions become increasingly complicated and uninviting to work with. Computational evidence leads us to make the following conjecture.

Conjecture 4.1. *For integers $j \geq 0, k \geq 2$, there is some N such that for all $n \geq N$ we have*

$$FD_{j,k}(n) \geq FO_{j,k}(n).$$

We use non-strict inequality to account for $k = 2$ but note that for $k > 2$ the inequality seems to become strict for n sufficiently large.

4.2. **Fibonacci Recurrence Relations.** Motivated by Straub's result (6) in [26], we prove via induction over n the following extensions, which give us as a corollary a Straub-type equality in Theorem 4.4.

Lemma 4.2. *For s a positive integer and $n > s + 1$,*

$$rd^{(s)}(n) := r(n \mid \textit{parts are distinct and smallest part} = s) = F_{n-s-1}.$$

Proof. Let s be fixed. The only partition counted by $rd^{(s)}(s + 2)$ is $(s + 1, s)$, and the only partition counted by $rd^{(s)}(s + 3)$ is $(s + 2, s)$, so the claim holds for $n = \{s + 2, s + 3\}$. Now suppose that $rd^{(s)}(n) = F_{n-s-1}$ holds for all valid choices of n up to $n = s + k$ with $k \geq 3$.

We observe that, in general, the set of partitions counted by $rd^{(s)}(s + r + 1)$ may be generated from the set of partitions counted by $rd^{(s)}(s + r)$ as follows.

Case A. The partitions π' counted by $rd^{(s)}(s + r + 1)$ that do not contain a part equal to $s + 1$ are in bijection with the set of partitions π counted by $rd^{(s)}(s + r)$ and may be generated by adding one to each part in π except for the part equal to s to obtain the unique corresponding π'.

Case B. The partitions π'' counted by $rd^{(s)}(s+r+1)$ that do contain a part equal to $s+1$ are in bijection with the set of partitions π' counted by $rd^{(s)}(s + r)$ that do not contain a part equal to $s + 1$ and may be generated by adding a part equal to $s + 1$ to obtain the unique corresponding π''. By Case A, we know these partitions π' may in turn be generated by the set of partitions π counted by $rd^{(s)}(s + r - 1)$.

Now, consider the partitions counted by $rd^{(s)}(s + k + 1)$. We observe that these partitions may be constructed from the partitions counted by $rd^{(s)}(s+k)$ and $rd^{(s)}(s+k-1)$ as follows.

- The set of partitions π' counted by $rd^{(s)}(s + k + 1)$ that do not contain a part equal to $s + 1$ are in bijection with the set of partitions π counted by $rd^{(s)}(s + k)$ by Case A above. By hypothesis, these are counted by F_{k-1}.
- The set of partitions π'' counted by $rd^{(s)}(s + k + 1)$ that do contain a part equal to $s + 1$ are in bijection with the set of partitions π' counted by $rd^{(s)}(s + k)$ that do not contain a part equal to $s + 1$ by Case B above, which are in turn in bijection with the set of partitions π counted by $rd^{(s)}(s + k - 1)$ by Case A. By hypothesis, these are counted by F_{k-2}.

The above cases account for all partitions counted by $rd^{(s)}(s+k+1)$ as it is clear each must either have a part of size $s+1$ or not have such a part. Then the number of partitions counted by $rd^{(s)}(s+k+1)$ is given by $F_{k-1}+F_{k-2}=F_k$, and the claim holds for $n=s+k+1$. By the principle of induction, the desired result follows. □

We apply similar reasoning to perimeter n partitions with odd parts.

Lemma 4.3. *For $s \geq 0$ and $n > 2s+1$,*

$$ro^{(s)}(n) := r(n \mid \textit{parts are odd and smallest part} = 2s+1) = F_{n-2s-1}.$$

Proof. Let s be fixed. It is evident that the only partition counted by $ro^{(s)}(2s+2)$ is $(2s+1, 2s+1)$ and that the only partition counted by $ro^{(s)}(2s+3)$ is $(2s+1, 2s+1), 2s+1$, so the claim holds for $n = \{2s+2, 2s+3\}$. Now suppose that $ro^{(s)}(n) = F_{n-2s-1}$ holds for all valid choices of n up to $2s+k$ with $k \geq 3$.

We observe that the set of partitions counted by $ro^{(s)}(2s+k+1)$ may be generated from the set of partitions counted by $ro^{(s)}(2s+k)$ and $ro^{(s)}(2s+k-1)$ as follows.

- The partitions π counted by $ro^{(s)}(2s+k+1)$ that at least 2 parts equal to $2s+1$ are in bijection with the partitions π' counted by $ro^{(s)}(2s+k)$ and may be generated by adding a new smallest part of $2s+1$ to each π' to obtain the unique corresponding π. By hypothesis, these are counted by F_{k-1}.
- The partitions π counted by $ro^{(s)}(2s+k+1)$ that contain exactly one part equal to $2s+1$ are in bijection with the partitions π'' counted by $ro^{(s)}(2s+k-1)$ and may be generated by adding 2 to each part in π' except for the part of size $2s+1$ to obtain the unique corresponding π. By hypothesis, these are counted by F_{k-2}.

Observe that the above cases account for all partitions counted by $rd^{(s)}(2s+k+1)$ as it is clear each must either have at least 2 parts of size $2s+1$ or exactly one such part. Then the number of partitions counted by $rd^{(s)}(2s+k+1)$ is given by $F_{k-1}+F_{k-2}=F_k$, and the claim holds for $n=2s+k+1$. By the principle of induction, the desired result follows. □

Lemmas 4.2 and 4.3 immediately yield the following theorem.

Theorem 4.4. *For s a positive integer and $n > s+1$,*

$$ro^{(s)}(n+s) = rd^{(s)}(n).$$

Remark 4.5. *Lemmas 4.2 and 4.3, and hence Theorem 4.4, can alternatively be proven using generating functions. We have that*

$$\sum_{n=s+2}^{\infty} rd^{(s)}(n)q^n = q^{s+1}\sum_{n=1}^{\infty} h_1(n)q^n = q^{s+1}\sum_{n=1}^{\infty} F_n q^n = \sum_{n=s+2}^{\infty} F_{n-s-1}q^n,$$

and

$$\sum_{n=2s+2}^{\infty} ro^{(s)}(n)q^n = q^{2s+1}\sum_{n=1}^{\infty} f_1(n)q^n = q^{2s+1}\sum_{n=1}^{\infty} F_n q^n = \sum_{n=2s+2}^{\infty} F_{n-2s-1}q^n,$$

where $h_1(n)$ (resp. $f_1(n)$) counts perimeter n partitions into distinct parts (resp. odd parts). Thus for $n > s+1$, $rd^{(s)}(n) = F_{n-s-1}$ and for $n > 2s+1$, $ro^{(s)}(n) = F_{n-2s-1}$.

4.3. **Analogue of Andrews's S-T Theorem.** An important theorem in the regular partition setting is Andrew's S-T result (9), due to its generality and applicability to many partition counting functions. In this section, we shall prove an analogue of this in the fixed perimeter setting through a counting proof and an injective proof. Recall that we have defined

$$S = \{a_0, a_1, a_2, \dots\}, \text{ where } a_{i+1} > a_i,$$
$$T = \{b_0, b_1, b_2, \dots\}, \text{ where } b_{i+1} > b_i,$$

with the added requirement that $a_i \geq b_i$ for all $i \in \mathbb{N}_0$. We prove that

$$r_T(n) = r(n \mid \text{parts in } T) \geq r(n \mid \text{parts in } S) = r_S(n).$$

Proof of Theorem 1.5. We begin by defining, for any $n \in \mathbb{N}$,

$$S_n = \{a_i \in S : a_i \leq n\},$$
$$T_n = \{b_i \in T : b_i \leq n\}.$$

For a given n, $r_S(n) = r_{S_n}(n)$ and $r_T(n) = r_{T_n}(n)$, as clearly no part in a partition of n can be larger than n. Since $b_i \leq a_i$ for all i, we have $|S_n| \leq |T_n|$ for all $n \in \mathbb{N}$ (i.e. if $S_n = \{a_0, a_1, \dots a_k\}$ then $\{b_0, b_1, \dots, b_k\} \subseteq T_n$). We can count $r_S(n)$ (resp. $r_T(n)$) by summing over the possible largest parts coming from S_n (resp. T_n):

$$r_S(n) = \sum_{\pi \in R_S(n)} r_S(\alpha, \lambda) = \sum_{i=0}^{|S_n|-1} r_S(a_i, n - a_i + 1),$$
$$r_T(n) = \sum_{\pi \in R_T(n)} r_T(\alpha, \lambda) = \sum_{i=0}^{|T_n|-1} r_T(b_i, n - b_i + 1).$$

We consider these on a term-by-term basis for a given $0 \leq i \leq |S_n| - 1$. We use a stars and bars counting technique, similar to the method used in [12, Lemma 4.3], to determine $r_S(\alpha, \lambda)$. For a given α, we determine the number of possible ways to choose the remaining $\lambda - 1$ parts (where here $\alpha = a_i$ and $\lambda = n - a_i + 1$). Since our largest part is a_i, we know the remaining parts can be chosen from the set S_{a_i}. We have that

$$r_S(a_i, \lambda) = \binom{|S_{a_i}| + \lambda - 2}{\lambda - 1}.$$

Since $|S_{a_i}| = i + 1$ and $\lambda = n - a_i + 1$, we have

$$r_S(a_i, n - a_i + 1) = \binom{n - a_i + i}{n - a_i},$$

and, similarly, for the same bounds on i, $0 \leq i \leq |S_n| - 1$ ($\leq |T_n| - 1$),

$$r_T(b_i, n - b_i + 1) = \binom{n - b_i + i}{n - b_i}.$$

Now, since we know that $a_i \geq b_i$, we can let $\delta_i = a_i - b_i \geq 0$. By substituting, we have that

$$\binom{n - b_i + i}{n - b_i} = \binom{n - a_i + i + \delta_i}{n - a_i + \delta_i} \geq \binom{n - a_i + i}{n - a_i},$$

with the inequality coming from the fact that $\binom{n+a}{k+a} \geq \binom{n}{k}$. As this is true for $0 \leq i \leq |S_n|-1$ and $|S_n| \leq |T_n|$, we have that

$$r_S(n) = \sum_{i=0}^{|S_n|-1} r_S(a_i, n-a_i+1) \leq \sum_{i=0}^{|T_n|-1} r_T(b_i, n-b_i+1) = r_T(n).$$

□

We now prove Theorem 1.5 through the use of an injection between the sets of partitions counted by $r_T(n)$ and $r_S(n)$.

Injective Proof of Theorem 1.5. We begin by considering the sets of partitions with parts in S and T respectively. We have $R_S(n) = \{\pi : \pi_i \in S\}$, $R_T(n) = \{\pi : \pi_i \in T\}$. Note that $|R_S(n)| = r_S(n)$ and $|R_T(n)| = r_T(n)$. Our desired result is to show that $|R_S(n)| \leq |R_T(n)|$.

Consider $\pi \in R_S(n)$ with $\pi = (\pi_1, \ldots, \pi_k)$. Note that each $\pi_i = a_j$ for some j. We will enumerate this as $\pi_i = a_{j_i}$. We then have that $n = a_{j_1} + k - 1$, as π_1 is the largest part and k tracks the number of parts. Also let $\delta i = a_i - b_i \geq 0$, and specifically let $x = \delta_{j_1}$. We define a function $f : R_S(n) \to R_T(n)$ where $f(\pi) = (f(\pi)_1, \ldots, f(\pi)_k, \ldots, f(\pi)_{k+x})$ and

$$f(\pi)_i = \begin{cases} b_{j_i}, & \text{if } 1 \leq i \leq k, \\ b_0, & \text{if } k < i \leq k+x. \end{cases}$$

Note that this preserves our perimeter n, as the x we are losing from the largest term is being added back to the perimeter by appending x parts to the end of the partition, i.e.,

$$n = a_{j_1} + k - 1 = (b_{j_1} + x) + k - 1 = b_{j_1} + (k+x) - 1. \tag{38}$$

We will let $B(n)$ represent the image of f, i.e., $B(n) = f(R_S(n))$. Note that $B(n) \subseteq R_T(n)$ as all of the parts of partitions in $B(n)$ come from T and the perimeter $\Gamma(f(\pi)) = n$ by (38). To remove confusion, we will let η denote partitions from $r_T(n)$, where $\eta = (\eta_1, \ldots, \eta_l)$. Similarly to before, we have that $\eta_i = b_m$ for some m. We will enumerate this such that $\eta_i = b_{m_i}$. Let $y = \delta_{m_1}$. Now, we can rewrite $B(n)$ as

$$B(n) = \{\lambda \in R_T(n) \mid \lambda_i = b_0, \text{ for } i > l - y\}.$$

It is thus apparent that $|B(n)| \leq |R_T(n)|$ so it remains to show that $|B(n)| = |R_S(n)|$. To do so, we construct an inverse $f^{-1} : B(n) \to R_S(n)$. We have for $\eta \in B(n)$ that $f^{-1}(\eta) = (f^{-1}(\eta)_1, \ldots, f^{-1}(\eta)_{l-y})$ where

$$f^{-1}(\eta)_i = a_{m_i} \text{ for } i \leq l - y.$$

We now show that this is, in fact, the inverse of f under its image. Let $\pi \in R_S(n)$, where $\pi = (a_{j_1}, \ldots, a_{j_k})$. We have that $f(\pi) = (b_{j_1}, \ldots, b_{j_k}, b_0, \ldots, b_0)$, where b_0 is repeated $x = \delta_{j_1}$ times. After re-indexing, we have that $f(\pi) = (b_{m_1}, \ldots, b_{m_l})$, where $l = k + x$ and $m_i = j_i$ for $i \leq k$, and otherwise $m_i = 0$. Now, since we must have $m_1 = j_1$, we know that $y = \delta_{m_1} = x$. We also know that $f(\pi) \in B(n)$, as $m_i = 0$ for $i > l - y = k$. Finally, we have that

$$f^{-1}(f(\pi)) = (a_{m_1}, \ldots, a_{m_k}) = (a_{j_1}, \ldots, a_{j_k}) = \pi.$$

We now have that $f : R_S(n) \to B(n)$ is a bijection, and since $B(n) \subseteq R_T(n)$, we are left with

$$r_S(n) = |R_S(n)| = |B(n)| \leq |R_T(n)| = r_T(n).$$

□

We now provide an example of the injection used in the previous proof, as this process is easier to understand by examining the diagram of a partition.

Example 4.6. In Figure 1, we display a visual example of the function $f(\pi)$ from Theorem 1.5. Here we have $S = \{2, 4, 6, 10, \ldots\}$ and $T = \{1, 3, 5, 7, \ldots\}$ with $\pi = (10, 6, 2, 2, 2)$ and $f(\pi) = \eta = (7, 5, 1, 1, 1, 1, 1, 1)$. Note that the perimeter $\Gamma(\pi) = \Gamma(\eta) = 14$.

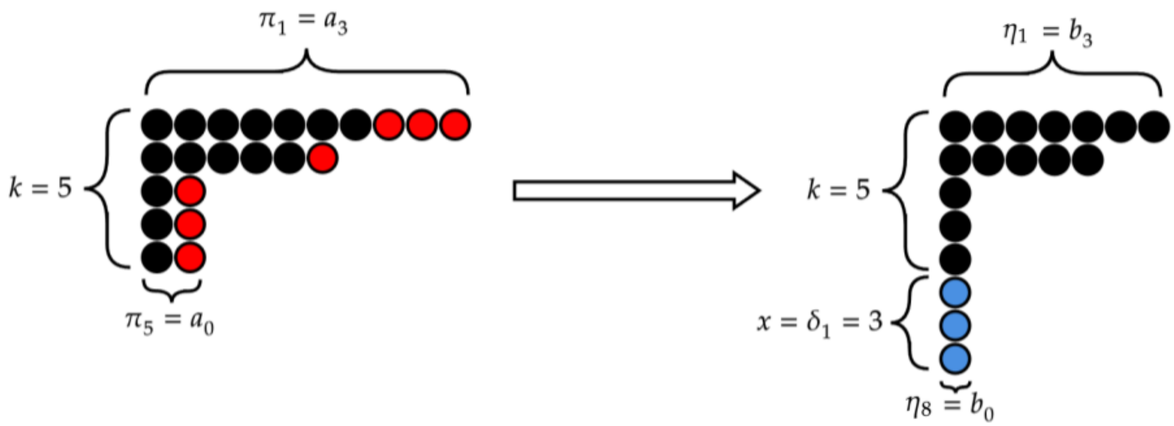


Figure 1. Example of Injection

Now that Theorem 1.5 has been proven, we shall first provide applications, and then propose extensions loosening the requirements on the sets S, T further.

Remark 4.7. *As in the regular partition setting, the existence of a fixed perimeter S-T theorem motivates some useful results, as well as simplified proofs of existing theorems. For instance, the inequalities* (12), (13), (14), (15) *given by* [12, Thm. 1.5, Prop. 1.6] *follow immediately from Theorem 1.5.*

We may adapt the injection f used to prove Theorem 1.5 for a more generalized version the same theorem. It is immediately clear that the requirement of $a_i \geq b_i$ for $i \geq 0$ may be changed to $i \geq 1$ in Theorem 1.5. Namely, if the largest part of a partition in the adapted $R_S(n)$ is equal to a_j for $j \geq 1$, then the same injection $f : R_S(n) \to R_T(n)$ works directly. If the largest part of such a partition is equal to a_0, then we adapt our mapping. In this case, the partition must consist entirely of parts equal to a_0, allowing us to map this directly to a partition of the same perimeter consisting of only parts equal to b_0 so long as $n \geq b_0$. We are motivated to make the following conjecture, which appears to hold from initial testing.

Conjecture 4.8. *Let $S = \{a_0, a_1, a_2, \dots\}$ and $T = \{b_0, b_1, b_2, \dots\}$ where $a_{i+1} > a_i$ and $b_{i+1} > b_i$ for $i \in \mathbb{N}_0$ as well as $a_i \geq b_i$ for $i \geq k$. Then there is some N such that for any $n \geq N$,*

$$r_T(n) \geq r_S(n).$$

When $k \geq 1$, the same injection used in the proof of Theorem 1.5 still holds for partitions counted by $r_S(n)$ with largest part equal to a_j, $j \geq k$. However, if the largest part is equal to a_j for $j < k$, adapting our injection becomes an unintuitive counting problem, as we must prove that there is ample room for these partitions to be mapped into $R_T(n)$.

4.4. **A Conjecture Concerning the Analogue of Kang-Kim Results.** Theorem 2.2 of the regular partition setting considers Alder-type inequalities while allowing the modulus and distinctness to change independently of each other. To work towards a similar result in the fixed perimeter we define the following function, generalizing $\ell_d^{(a)}(n)$ defined by Chen et al. [12]. Note that we have changed notation from Chen et al. by shifting the modulus from $d+3$ to d to allow for cleaner proofs. Let a, b, d, n positive integers and $0 < a < b \leq d$ and $n \geq 0$.

$$\ell_d^{(a,b)}(n) := r(n \mid \text{ parts are } \equiv a, b \pmod{d}).$$

We also define the following refinement.

$$\begin{aligned}\ell_d^{(a,b)}(\alpha,\lambda) &= \ell_d^{(a,b)}(\alpha,\lambda,n)\\ &= r(n \mid \text{ parts are } \equiv a, b \pmod{d} \text{ with largest part } \alpha \text{ and } \lambda \text{ parts}).\end{aligned}$$

We further define $\mathcal{L}_d^{(a,b)}(n)$ to be the set of partitions counted by $\ell_d^{(a,b)}(n)$. Recall from the introduction that we have

$$\begin{aligned}h_d^{(a)}(n) &:= r(n \mid \text{ parts are d-distinct and } \geq a),\\ f_d^{(a)}(n) &:= r(n \mid \text{ parts are } \equiv a (\text{mod } d+1)).\end{aligned}$$

To consider an analogue for Theorem 2.2, we compare $h_d^{(a)}(n)$ and $\ell_d^{(a,b)}(n)$. We make the following conjecture.

Conjecture 4.9. *For positive integers $0 < m_1 < m_2 \leq m$, $0 < a \leq d$,*

$$\lim_{n\to\infty}\left(h_d^{(a)}(n) - \ell_m^{(m_1,m_2)}(n)\right) = \begin{cases} +\infty & m > 2(d+1),\\ -\infty & m < 2(d+1).\end{cases}$$

Furthermore if $m = 2(d+1)$, then

$$\lim_{n\to\infty}\left(h_d^{(a)}(n) - \ell_m^{(m_1,m_2)}(n)\right) = \begin{cases} 0 & m_1 = a, m_2 = a+(d+1),\\ +\infty & m_1 \geq a, m_2 \geq a+(d+1), \text{ not both equal},\\ -\infty & m_1 \leq a, m_2 \leq a+(d+1), \text{ not both equal}.\end{cases}$$

When $m_1 > a, m_2 < a+(d+1)$ or $m_1 < a, m_2 > a+(d+1)$, the result is unclear from our experimentation.

Remark 4.10. *A first insight into proving this conjecture comes from the result* (8)*, stating that* $h_d^{(a)}(n) = f_d^{(a)}(n)$*. Since* $f_d^{(a)}(n)$ *considers parts equivalent to* a *modulo* $d+1$*, we can instead consider parts equivalent to* $a, a+d+1$ *modulo* $2(d+1)$*, giving us that* $h_d^{(a)}(n) = f_d^{(a)}(n) = \ell_{2(d+1)}^{(a,a+d+1)}(n)$*. Thus, our problem reduces to proving shift inequalities for a general* $\ell_d^{(a,b)}(n)$*.*

From here, the second set of equalities, for the case when $m = 2(d+1)$*, becomes quite simple to prove. Let* $a', b' \in \mathbb{N}$ *such that* $a' \geq a$ *and* $b' \geq b$*. We can apply Theorem 1.5 to have*

$$\ell_d^{(a,b)}(n) \geq \ell_d^{(a',b')}(n).$$

Applying this to $h_d^{(a)} = \ell_{2(d+1)}^{(a,a+d+1)}(n)$ *and* $\ell_m^{(m_1,m_2)}$ *immediately gives us our second set of equalities.*

The rest of the conjecture becomes much more tedious to prove, but we provide the following lemmas that may be useful in future research.

Lemma 4.11. *For positive integers* d, a, b *with* $a < b \leq d$*, we have*

$$\mathscr{L}_d^{(a,b)}(x,y,q) := \sum_{\lambda=0}^{\infty}\sum_{\alpha=0}^{\infty} \ell_d^{(a,b)}(\alpha,\lambda) x^\alpha y^\lambda q^{\alpha+\lambda-1} = \frac{x^a y q^a \left(1 + \frac{x^{b-a}q^{b-a}}{1-yq}\right)}{(1-yq)\left(1 - \frac{x^d q^d}{(1-yq)^2}\right)} = \frac{x^b y q^b + x^a y q^a (1-yq)}{(1-yq)^2 - x^d q^d}.$$

Proof. To construct this generating function to count all $\pi \in \mathcal{L}_d^{(a,b)}(x,y,q)$, we shall work along such a partitions profile given by w_π. We refer readers the beginning of Section 4 and the generating function proof in Section 4.1 for a complete description of this process. Each part π_i of such a partition π must be congruent to a or b modulo d, and, as such, the total number of Es in our word must be congruent to a or b modulo d. We can split up w_π into the following blocks, then encode them into a generating function. In each of the following, $j \in \mathbb{N}_0$.

- Initialization: Our first a consecutive Es followed by j Ns.
- Type I: $b-a$ consecutive Es followed by j Ns.
- Type II: $d-b+a$ consecutive Es followed by j Ns.
- Finalization: A single N to complete our profile.

The structure of our word will always begin with initialization and conclude with finalization. Between these, the body of the partition will alternate between Type I and Type II as many or as few times as necessary. As we explain next, the body (if nonempty) will always begin with a Type I block, then begin alternating between Type II and Type I. We next translate each of these blocks into the terms of a partition rather than our word notation.

- Initialization requires our partition to be in $\mathcal{L}_d^{(a,b)}$ by ensuring the smallest part is $\geq a$, allowing for j inclusions of this part.
- Type I takes a part size from being $\equiv a \pmod d$ to the next part size $\equiv b \pmod d$, then allowing for j inclusions of this part.

- Type II takes a part size from being $\equiv b \pmod{d}$ to the next part size $\equiv a \pmod{d}$, then allowing for j inclusions of this part.
- Finalization ends our profile, ensuring we include ≥ 1 part of the current size.

As the initial step brings our current part size to being $\equiv a \pmod{d}$, it must be followed by a Type I block, if the body is nonempty. We now convert these into generating functions.

- Initialization: $x^a q^a(1 + yq + y^2q^2 + \cdots) = \frac{x^a q^a}{1-yq}$
- Type I: $x^{b-a}q^{b-a}(1 + yq + y^2q^2 + \cdots) = \frac{x^{b-a}q^{b-a}}{1-yq}$
- Type II: $x^{d-b+a}q^{d-b+a}(1 + yq + y^2q^2 + \cdots) = \frac{x^{d-b+a}q^{d-b+a}}{1-yq}$
- Finalization: y

We omit a q for finalization to ensure that $n = \alpha + \lambda - 1$. Together, we have

$$\begin{aligned}
&\mathscr{L}_d^{(a,b)}(x, y, q)\\
&= \frac{x^a y q^a}{1-yq}\left(1 + \frac{x^{b-a}q^{b-a}}{1-yq} + \frac{x^{b-a}q^{b-a}}{1-yq}\frac{x^{d-b+a}q^{d-b+a}}{1-yq} + \frac{x^{b-a}q^{b-a}}{1-yq}\frac{x^{d-b+a}q^{d-b+a}}{1-yq}\frac{x^{b-a}q^{b-a}}{1-yq} + \cdots\right)\\
&= \frac{x^a y q^a}{1-yq}\left(\frac{1 + \frac{x^{b-a}q^{b-a}}{1-yq}}{1 - \frac{x^{d-b+a}q^{d-b+a}}{1-yq}\frac{x^{b-a}q^{b-a}}{1-yq}}\right)\\
&= \frac{x^a y q^a\left(1 + \frac{x^{b-a}q^{b-a}}{1-yq}\right)}{(1-yq)\left(1 - \frac{x^d q^d}{(1-yq)^2}\right)}.
\end{aligned}$$

□

We also provide an explicit count for $\ell_d^{(a,b)}(n)$, given by the following lemma.

Lemma 4.12. *For integers $a < b \leq d$ and $n \geq 0$ we have that*

$$\ell_d^{(a,b)}(n) = \sum_{k=0}^{\lfloor \frac{n-a}{d} \rfloor} \binom{n-a-dk+2k}{n-a-dk} + \sum_{k=0}^{\lfloor \frac{n-b}{d} \rfloor} \binom{n-b-dk+2k+1}{n-b-dk}.$$

Proof. As in the first proof of Section 4.3 and in [12, Pf. of Lemma 4.3], we shall sum over the possible largest parts for partitions in $\mathcal{L}_d^{(a,b)}(n)$. We have

$$\begin{aligned}
\ell_d^{(a,b)}(n) &= \sum_{\substack{1\leq\alpha\leq n\\ \alpha\equiv a\ (d)}} \ell_d^{(a,b)}(\alpha, n-\alpha+1)\\
&= \sum_{\substack{1\leq\alpha\leq n\\ \alpha\equiv a,b\ (d)}} \ell_d^{(a,b)}(\alpha, n-\alpha+1) + \sum_{\substack{1\leq\alpha\leq n\\ \alpha\equiv b\ (d)}} \ell_d^{(a,b)}(\alpha, n-\alpha+1)\\
&= \sum_{k=0}^{\lfloor \frac{n-a}{d} \rfloor} \ell_d^{(a,b)}(a+dk, n-a-dk+1) + \sum_{k=0}^{\lfloor \frac{n-b}{d} \rfloor} \ell_d^{(a,b)}(b+dk, n-b-dk+1).
\end{aligned} \tag{39}$$

We shall determine the value of $\ell_d^{(a,b)}(\alpha, \lambda)$ for a given α, λ using a stars and bars counting technique. We know by definition of $\ell_d^{(a,b)}(n)$ that we must have $\alpha \equiv a, b \pmod{d}$. With α

fixed, the remaining $\lambda - 1$ parts must be chosen from the set

$$S_\alpha = \{1 \leq x \leq \alpha \mid x \equiv a, b \pmod{d}\}.$$

Thus,

$$\ell_d^{(a,b)}(\alpha, \lambda) = \binom{|S_\alpha| + \lambda - 2}{\lambda - 1}. \tag{40}$$

We shall now determine $|S_\alpha|$. We have

$$\begin{aligned} S_\alpha &= \{1 \leq a + dj \leq \alpha \mid j \in \mathbb{Z}\} \sqcup \{1 \leq b + dj \leq \alpha \mid j \in \mathbb{Z}\} \\ &= \{a + dj \mid 0 \leq j \leq \lfloor \tfrac{\alpha - a}{d} \rfloor\} \sqcup \{b + dj \mid 0 \leq j \leq \lfloor \tfrac{\alpha - b}{d} \rfloor\}. \end{aligned}$$

This gives us

$$|S_\alpha| = \lfloor \tfrac{\alpha - a}{d} \rfloor + \lfloor \tfrac{\alpha - b}{d} \rfloor + 2 = \begin{cases} 2\left(\frac{\alpha - a}{d}\right) + 1 & \alpha \equiv a \pmod{d}, \\ 2\left(\frac{\alpha - b}{d}\right) + 2 & \alpha \equiv b \pmod{d}. \end{cases} \tag{41}$$

Plugging (41) into (40) gives

$$\ell_d^{(a,b)}(\alpha, \lambda) = \begin{cases} \dbinom{2\left(\frac{\alpha - a}{d}\right) + \lambda - 1}{\lambda - 1} & \alpha \equiv a \pmod{d}, \\ \dbinom{2\left(\frac{\alpha - b}{d}\right) + \lambda}{\lambda - 1} & \alpha \equiv b \pmod{d}. \end{cases}$$

We now return to (39) to arrive at our desired result,

$$\begin{aligned} \ell_d^{(a,b)}(n) &= \sum_{k=0}^{\lfloor \frac{n-a}{d} \rfloor} \binom{2k + (n - a - dk + 1) - 1}{n - a - dk + 1 - 1} + \sum_{k=0}^{\lfloor \frac{n-b}{d} \rfloor} \binom{2k + (n - b - dk + 1)}{n - b - dk + 1 - 1} \\ &= \sum_{k=0}^{\lfloor \frac{n-a}{d} \rfloor} \binom{n - a - dk + 2k}{n - a - dk} + \sum_{k=0}^{\lfloor \frac{n-b}{d} \rfloor} \binom{n - b - dk + 2k + 1}{n - b - dk}. \end{aligned}$$

□

4.5. **A Beck-Type Companion Identity for a Result of Chen et. al.** Motivated by the recent developments allowing for Beck-type companions to be derived for many partition counting identities, we sought a companion for the identity of Chen et al. [12, Thm. 1.4] given by

$$f_d^{(a)}(n) = h_d^{(a)}(n).$$

We shall first create a generalized definition to denote the excess in number of parts, as was used in Section 3.2. For partition counting functions $a(n)$ and $b(n)$, we define an operator E,

$$E(a(n), b(n)) \coloneqq \sum_{\pi \in A(n)} \ell(\pi) - \sum_{\pi \in B(n)} \ell(\pi),$$

where $\ell(\pi)$ is used to count the number of parts of a partition π. Furthermore, if $\mathcal{A}(q)$ (resp. $\mathcal{B}(q)$) is the generating function for $a(n)$ (resp. $b(n)$), then we have

$$E(\mathcal{A}(q), \mathcal{B}(q)) \coloneqq \sum_{n=0}^{\infty} E(a(n), b(n))q^n.$$

In the following lemma, we derive the generating function for $E(f_d^{(a)}(n), h_d^{(a)}(n))$.

Lemma 4.13. *For positive integers n, d, a with $a \leq d$, we have that*

$$E(\mathcal{F}_d^{(a)}(q), \mathcal{H}_d^{(a)}(q)) = \sum_{n=0}^{\infty} E(f_d^{(a)}(n), h_d^{(a)}(n))q^n = \frac{q^{a+1} - q^{a+d+1}}{(1-q-q^{d+1})^2}.$$

Proof. By Chen et al. [12], we have the following refined generating functions for $f_d^{(a)}(n)$ and $h_d^{(a)}(n)$.

$$\mathcal{F}_d^{(a)}(x, y, q) = \sum_{\lambda=0}^{\infty}\sum_{\alpha=0}^{\infty} f_d^{(a)}(\alpha, \lambda) x^{\alpha} y^{\lambda} q^{\alpha+\lambda-1} = \frac{x^a y q^a}{1 - yq - x^{d+1}q^{d+1}},$$

$$\mathcal{H}_d^{(a)}(x, y, q) = \sum_{\lambda=0}^{\infty}\sum_{\alpha=0}^{\infty} h_d^{(a)}(\alpha, \lambda) x^{\alpha} y^{\lambda} q^{\alpha+\lambda-1} = \frac{x^a y q^a}{1 - xq - x^{d}yq^{d+1}}.$$

Recall that in these formulas λ refers to the leg length, or number of parts within a given partition. In order to find $E(\mathcal{F}_d^{(a)}(q), \mathcal{H}_d^{(a)}(q))$ we can differentiate with respect to y to count λ for each partition, then set $x = y = 1$. We get

$$\begin{aligned}\frac{\partial}{\partial y}\bigg|_{x=y=1} \mathcal{F}_d^{(a)}(x, y, q) &= \frac{x^a q^a (1 - xq - x^d y q^{d+1}) - x^a y q^a(-q)}{(1 - yq - x^{d+1}q^{d+1})^2}\bigg|_{x=y=1} \\ &= \frac{q^a - q^{a+1} - q^{a+d+1} + q^{a+1}}{(1-q-q^{d+1})^2} = \frac{q^a - q^{a+d+1}}{(1-q-q^{d+1})^2}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial y}\bigg|_{x=y=1} \mathcal{H}_d^{(a)}(x, y, q) &= \frac{x^a q^a (1 - xq - x^d y q^{d+1}) - x^a y q^a(-x^d q^{d+1})}{(1 - xq - x^{d}yq^{d+1})^2}\bigg|_{x=y=1} \\ &= \frac{q^a - q^{a+1} - q^{a+d+1} + q^{a+d+1}}{(1-q-q^{d+1})^2} = \frac{q^a - q^{a+1}}{(1-q-q^{d+1})^2}.\end{aligned}$$

To find $E(\mathcal{F}_d^{(a)}(q), \mathcal{H}_d^{(a)}(q))$, we take the difference of the above terms. We then have

$$\begin{aligned}E(\mathcal{F}_d^{(a)}(q), \mathcal{H}_d^{(a)}(q)) &= \frac{q^a - q^{a+d+1}}{(1-q-q^{d+1})^2} - \frac{q^a - q^{a+1}}{(1-q-q^{d+1})^2} \\ &= \frac{q^{a+1} - q^{a+d+1}}{(1-q-q^{d+1})^2}.\end{aligned}$$

□

We now construct a function which will provide our Beck-type identity. For positive integers n, d, a with $d \geq 2$, $a \leq d$, we let $G_d^{(a)}(n)$ denote the set of partitions $\pi = (\pi_1, \dots, \pi_r)$ with perimeter $\Gamma(\pi) = n$ that are constructed by the following algorithm.

(i) Let $\pi_{r+1} := a$,

(ii) $\pi_i = \begin{cases} \pi_{i+1} \text{ or } \pi_{i+1} + d \text{ or } \pi_{i+1} + (d-1), & \text{if } \pi_{i+1} \equiv a \pmod{d} \\ \pi_{i+1} \text{ or } \pi_{i+1} + d, & \text{if } \pi_{i+1} \equiv a - 1 \pmod{d} \end{cases}$ for $2 \leq i \leq r$,

(iii) π_1 bounded by $\lfloor \frac{\pi_2}{d} \rfloor d + a < \pi_1 \leqslant \lfloor \frac{\pi_2}{d} \rfloor d + a + d$.

We further let $g_d^{(a)}(n)$ denote $|G_d^{(a)}(n)|$. Alternatively,
$g_d^{(a)}(n) = r(n \mid$

(a) largest part occurs exactly once

(b) all other parts are $\equiv a, a-1 \pmod{d}$ and differ by at most d

(c) smallest part $= a$, $a + d$, or $a + d - 1$

(d) body of partition can be split into exactly two (possibly empty) sections containing only parts $\equiv a \pmod{d}$ or only parts $\equiv a - 1 \pmod{d}$, with the section containing parts $\equiv a - 1 \pmod{d}$ closer to the top

(e) if section containing parts $\equiv a - 1 \pmod{d}$ is nonempty, then largest part differs from neighbor by ≥ 2 and $\leq d + 1$, else largest part differs from neighbor by ≥ 1 and $\leq d$).

Theorem 4.14. *For positive integers n, d, a with $d \geq 2$ and $a \leq d$, we have*

$$\mathcal{G}_d^{(a)}(x, y, q) = \sum_{\lambda=0}^{\infty} \sum_{\alpha=0}^{\infty} g_d^{(a)}(\alpha, \lambda) x^\alpha y^\lambda q^{\alpha+\lambda-1} = \frac{x^{a+1} y q^{a+1}(1 - x^d q^d)}{(1 - yq)(1 - xq)(1 - \frac{x^d y q^{d+1}}{1 - yq})^2}. \tag{42}$$

In particular,

$$E(f_d^{(a)}(n), h_d^{(a)}(n)) = g_d^{(a)}(n). \tag{43}$$

Proof. By taking $x = y = 1$ in (42), we have

$$\mathcal{G}_d^{(a)}(1, 1, q) = \sum_{n=1}^{\infty} g_d^{(a)}(n) q^n = \frac{q^{a+1}(1 - q^d)}{(1 - q)^2 (1 - \frac{q^{d+1}}{1 - q})^2}$$
$$= \frac{q^{a+1} - q^{a+d+1}}{(1 - q - q^{d+1})^2} = \sum_{n=1}^{\infty} E(f_d^{(a)}(n), h_d^{(a)}(n)) q^n$$

by Lemma 4.13. So we are left with (43).

To prove (42), we will interpret the definition and then translate it into a generating function. We begin with the smallest part π_r of the partition and build our way up to the largest part π_1. The body of the partition, described by Condition (ii), can be split into two sections; Section A consisting of parts equivalent to a modulo d and Section B consisting of parts equivalent to $a - 1$ modulo d. We claim Condition (ii) requires that the body of the partition consists of Section A followed by Section B, as is given by **(d)**.

Considering some part π_i in the body of the partition, we will examine the different cases of Condition (ii). The first case considers when $\pi_{i-1} \equiv a \pmod{d}$. Then π_i may either remain in Section A ($\pi_i = \pi_{i+1}$ or $\pi_{i+1} + d$) or swap to Section B ($\pi_i = \pi_{i+1} + (d - 1)$). The second case of (ii) considers when $\pi_{i-1} \equiv a-1 \pmod{d}$. Here π_i is required to remain in Section B ($\pi_i = \pi_{i+1}$ or $\pi_{i+1}+d$). Then our claim is satisfied as Section B cannot precede Section A.

Condition (i) initializes our algorithm and ensures that $\pi_r = a$, $a+d$, or $a+d-1$, as stated by **(c)**. Condition (iii) finalizes the algorithm and determines the possible options for π_i. To do so, it finds smallest integer $\equiv a(\text{mod } d)$ and $\geq \pi_2$ and allows π_1 to be drawn from the next d integers. For example, if $\pi_2 = a+dk$ then $\pi_1 = a+dk+1, a+dk+2, \ldots, a+dk+d$.

Observe that Sections A, B may be empty. If $\pi_r = a+d-1$, then we surpass Section A as our smallest part already lies in Section B. If π_2 is in Section A, then we surpass Section B and go to the finalization term.

Now, we will translate this into a generating function split into four sections: (I)nitialization, Section (A), Section (B), and (F)inalization. We will consider the profile taken on by the partition through each of these sections, first by describing the possible routes outlined by our conditions on $\pi \in G_d^{(a)}(n)$. We have the following possibilities.

- IABF
- IAF
- IBF
- IF

Note that each route must begin with initialization and end with finalization. We encode each section individually but will combine them in our generating function.

Initialization: We encode Condition (i) by beginning our profile with a steps E and no steps N, to simulate the previous part (π_{r+1}) being equal to a. We can now either repeat this part $j \geq 0$ times or take d (resp $d-1$) steps E. The latter option is encoded in our generating function for Section A (resp. B). So allowing for j steps N, initialization takes the form

$$x^a q^a (1 + yq + (yq)^2 + (yq)^3 + \cdots) = \frac{x^a q^a}{1-yq}.$$

Section A: Traversing our profile through Section A, we can either take a single step N ($\pi_i = \pi_{i+1}$) or take d steps E followed by a single step N ($\pi_i = \pi_{i+1}+d$). These are encoded by yq and $x^d q^{d+1} y$, respectively. We can reinterpret this by considering blocks based on when $\pi_{i+1}+d$ is chosen and allowing for $j \geq 0$ steps N within each block. Each of these blocks has the following format.

$$x^d y q^{d+1} (1 + yq + (yq)^2 + (yq)^3 + \cdots) = \frac{x^d y q^{d+1}}{1-yq}.$$

Now, Section A in total will contain $k \geq 0$ of these blocks. So, section A has the format

$$1 + \frac{x^d y q^{d+1}}{1-yq} + (\frac{x^d y q^{d+1}}{1-yq})^2 + \cdots = \left(1 - \frac{x^d y q^{d+1}}{1-yq}\right)^{-1}.$$

Section B: Traversing our profile through Section B takes on almost the same form as Section A, except that we must include the shift from A into B. To do so, we will prepend a new block which encodes the option $\pi_i = \pi_{i+1} + (d-1)$. As this shift can only occur once in the partition, we need only include it a single time. We will also allow for the part of this size to be repeated $j \geq 0$ times, or j steps N. So, our new block has the format

$$\frac{x^{d-1}yq^d}{1-yq},$$

and overall Section B has the format

$$\frac{x^{d-1}yq^d}{1-yq}\left(1-\frac{x^d yq^{d+1}}{1-yq}\right)^{-1}.$$

Finalization: To determine our final part, we first find the smallest integer $\geq \pi_2$ that is equivalent to a modulo d, allowing for moves E until the current arm length is equivalent to a modulo d. If we move from A to F or I to F, then our arm length is already equivalent to a modulo d. If we move from B to F, then we need one move E to become equivalent to a modulo d. We will denote these cases F_A and F_B respectively, and we will first consider F_A then adapt it to F_B by prepending a move E.

If our current arm length is equivalent to a modulo d, we need to allow for π_1 to be drawn from the next d integers, so we allow for k moves E where $1 \leq k \leq d$. Since finalization includes exactly one part, we include one move N after our k moves E. Now, our F_A takes on the format

$$y(xq + x^2q^2 + \cdots + x^dq^d) = \frac{xyq(1-(xq)^d)}{1-xq}.$$

In the above, we do not include a q for the move N to ensure that in our final generating function $n = \alpha + \lambda - 1$. We obtain a generating function for F_B by prepending a move E, so our format is

$$xq * y(xq + x^2q^2 + \cdots + x^dq^d) = xq\frac{xyq(1-(xq)^d)}{1-xq}.$$

Now that we have encoded each of our sections, we put them together in such a way that allows for every possible route. Our generating function for $g_d^{(a)}(n)$ will take the form

$$\mathrm{IA}(\mathrm{F_A} + \mathrm{B}(\mathrm{F_B})).$$

Note that since Section $\mathrm{A} = 1 + \frac{x^dq^{d+1}y}{1-yq} + \cdots$, our profile can pass through section A unchanged by multiplying through the 1. We now substitute each of these sections in to obtain our complete generating function.

$$\begin{aligned}
&\mathcal{G}_d^{(a)}(x,y,q) \\
&\quad= \left(\frac{x^aq^a}{1-yq}\right)\left(1-\frac{x^dyq^{d+1}}{1-yq}\right)^{-1} \\
&\quad\cdot\left[\frac{xyq(1-(xq)^d)}{1-xq} + \left(\frac{x^{d-1}yq^d}{1-yq}\left(1-\frac{x^dyq^{d+1}}{1-yq}\right)^{-1}\right)\left(xq\frac{xyq(1-(xq)^d)}{1-xq}\right)\right] \\
&\quad= \left(\frac{x^aq^a}{1-yq}\right)\left(1-\frac{x^dyq^{d+1}}{1-yq}\right)^{-1}\left[1+\frac{x^dyq^{d+1}}{1-yq}\left(1-\frac{x^dyq^{d+1}}{1-yq}\right)^{-1}\right]\left(\frac{xyq(1-(xq)^d)}{1-xq}\right)
\end{aligned}$$

$$= \left(\frac{x^a q^a}{1-yq}\right)\left(1-\frac{x^d y q^{d+1}}{1-yq}\right)^{-1}\left[\left(1-\frac{x^d y q^{d+1}}{1-yq}\right)^{-1}\right]\left(\frac{xyq(1-(xq)^d)}{1-xq}\right)$$

$$= \frac{x^{a+1} y q^{a+1}(1-x^d q^d)}{(1-yq)(1-xq)(1-\frac{x^d y q^{d+1}}{1-yq})^2}.$$

□

5. Fixed Perimeter Analogues of Parity Inequalities in Partitions

Recall that $r_o(n)$ (resp. $r_e(n)$) is the number of perimeter n partitions having more odd parts than even parts (resp. more even parts than odd parts). We will use $\mathcal{R}$ in place of r to denote the set of partitions counted by a particular partition counting function (for instance, $\mathcal{R}_e(n)$ denotes the set of partitions counted by $r_e(n)$). Throughout this section, let

$$\omega(\pi) := (\text{number of odd parts in } \pi) - (\text{number of even parts in } \pi).$$

5.1. **Parity Bias for Partitions with Restriction of Allowed Parts.** Let $r^S(n)$ denote the number of perimeter n partitions with no parts coming from S. Recall that we use S_0 to denote the empty set and S_k to denote the set of integers $\{1, 2, ..., k\}$. In the fixed perimeter setting, we find that regular partition parity bias results can be generalized through an injection technique, with slightly lower bounds required on n. In particular, we have below that the fixed perimeter analogues of (16) and (18) are special cases given by $k = 0$ in (45) and $k = 1$ in (44), respectively, and Theorem 1.7 is obtained by taking $k = 0$ in (44).

Theorem 5.1. *Let $k \in \mathbb{N}_0$. For all $n = 2k+1$ and $n \geq 2k+3$,*

$$r_o^{\{S_{2k}\}}(n) > r_e^{\{S_{2k}\}}(n), \tag{44}$$

and for all $n = 2k+2$ and $n \geq 2k+4$,

$$r_o^{\{S_{2k+1}\}}(n) < r_e^{\{S_{2k+1}\}}(n). \tag{45}$$

Proof. We begin by considering (44). The result is clear when $n = 2k+1$ as the only allowable partition is $\pi = (n)$. The following mapping applies to partitions of perimeter $n \geq 2k+3$. Consider a partition $\pi \in \mathcal{R}_e^{\{S_{2k}\}}(n)$ and define an injective mapping by

$$\Psi_1 = \varphi_1 \cup \psi_1 : \mathcal{R}_e^{\{S_{2k}\}}(n) \to \mathcal{R}_o^{\{S_{2k}\}}(n).$$

(1) If π contains at least one part equal to $2k+1$, map π to π'' via φ_1 with π' as an intermediary step. First, delete the smallest part (of size $2k+1$). The resulting partition $\pi' \in \mathcal{R}_e^{\{S_{2k}\}}(n-1)$ as we have only removed an odd part. Now add 1 to each part of π' to form π''. The resulting partition π'' has perimeter n with no parts less than or equal to $2k$ and reverses the parity of each part in π', thus $\pi'' \in \mathcal{R}_o^{\{S_{2k}\}}(n)$.
(2) If π contains no parts equal to $2k+1$, map π to π'' via ψ_1 with π' as an intermediary step. First, subtract 1 from every part. The resulting partition π' has perimeter $n-1$, reverses the parity of each part of π, and has minimum possible part size $2k+1$, thus $\pi' \in \mathcal{R}_o^{\{S_{2k}\}}(n-1)$. Now, append a new part of size $2k+1$ to π'. The resulting partition π'' has perimeter n, thus $\pi'' \in \mathcal{R}_o^{\{S_{2k}\}}(n)$ as we have only added an odd part of size $2k+1$.

There exists a mapping for every $\pi \in \mathcal{R}_e^{\{S_{2k}\}}(n)$ since each $\pi \in \mathcal{R}_e^{\{S_{2k}\}}(n)$ either has a part of size $2k+1$ or does not have a part of size $2k+1$. We now introduce the following sets.

$$
\begin{aligned}
&E_a(n) = \{\pi \in \mathcal{R}_e^{\{S_a\}}(n) \mid \pi \text{ has at least one part equal to } a+1\},\\
&E_a^c(n) = \{\pi \in \mathcal{R}_e^{\{S_a\}}(n) \mid \pi \text{ has no parts equal to } a+1\},\\
&\widetilde{E}(n) = \{\pi \in \mathcal{R}_e^{\{S_a\}}(n) \mid \omega(\pi) \le -2\},\\
&O_a(n) = \{\pi \in \mathcal{R}_o^{\{S_a\}}(n) \mid \pi \text{ has at least one part equal to } a+1\},\\
&O_a^c(n) = \{\pi \in \mathcal{R}_o^{\{S_a\}}(n) \mid \pi \text{ has no parts equal to } a+1\},\\
&\widetilde{O}(n) = \{\pi \in \mathcal{R}_o^{\{S_a\}}(n) \mid \omega(\pi) \ge 2\},\\
&\widetilde{O}_a(n) = \widetilde{O}(n) \cap O_a(n) \text{ and } \widetilde{O}_a^c(n) = \widetilde{O}(n) \cap O_a^c(n),\\
&\widetilde{E}_a(n) = \widetilde{E}(n) \cap E_a(n) \text{ and } \widetilde{E}_a^c(n) = \widetilde{E}(n) \cap E_a^c(n).
\end{aligned}
$$

Note that fixing $a = 2k$, $\mathcal{R}_e^{\{S_{2k}\}}(n) = E_{2k}(n) \sqcup E_{2k}^c(n)$ and $\widetilde{O}(n) = \widetilde{O}_{2k}(n) \sqcup \widetilde{O}_{2k}^c(n)$. We have that φ_1 is a mapping from $E_{2k}(n)$ to $\widetilde{O}_{2k}^c(n)$ and ψ_1 is a mapping from $E_{2k}^c(n)$ to $\widetilde{O}_{2k}(n)$. Both mappings are invertible: given $\pi'' \in \widetilde{O}_{2k+1}^c(n)$, the unique corresponding $\pi \in E_{2k}(n)$ is found by by subtracting 1 from every part in π'' and adding a new smallest part of $2k$. Likewise, given $\pi'' \in \widetilde{O}_{2k}(n)$, the unique corresponding $\pi \in E_{2k}^c(n)$ is found by deleting the smallest part $2k$ and adding one to every part.

So φ_1 and ψ_1 are bijections and thus Ψ_1 is a bijection from $\mathcal{R}_e^{\{S_{2k}\}}(n)$ to $\widetilde{O}(n) \subsetneq \mathcal{R}_o^{\{S_{2k}\}}(n)$. The strict subset follows from the fact that for any choice of n we have $\mathcal{R}_o^{\{S_{2k}\}}(n) \setminus \widetilde{O}(n) \neq \emptyset$. (i.e., if $n \ge 2k+3$ odd then $\pi = (n) \in \mathcal{R}_o^{\{S_{2k}\}}(n) \setminus \widetilde{O}(n)$, and if $n \ge 2k+4$ even then $\pi = (n-2, 2k+1, 2k+1) \in \mathcal{R}_o^{\{S_{2k}\}}(n) \setminus \widetilde{O}(n)$). We are left with (44),

$$r_e^{\{S_{2k}\}}(n) = |\mathcal{R}_e^{\{S_{2k}\}}(n)| = |\widetilde{O}(n)| < |\mathcal{R}_o^{\{S_{2k}\}}(n)| = r_o^{\{S_{2k}\}}(n).$$

We adapt this process and apply it to (45). The result is clear when $n = 2k+1$ as the only allowable partition is $\pi = (n)$, so we will suppose $n \ge 2k+3$. Consider a partition $\pi \in \mathcal{R}_o^{\{S_{2k+1}\}}(n)$ and define an injective mapping

$$\Psi_2 = \psi_2 \cup \varphi_2 : \mathcal{R}_o^{\{S_{2k+1}\}}(n) \to \mathcal{R}_e^{\{S_{2k+1}\}}(n)$$

as follows.

(1) If π contains at least one part equal to $2k+2$, map π to π'' via φ_2 with π' as an intermediary step. First, delete the smallest part (of size $2k+2$). The resulting $\pi' \in \mathcal{R}_o^{\{S_{2k+1}\}}(n-1)$ as we have only removed an even part. Now, add 1 to each part of π'. The resulting π'' has perimeter n with no parts less than or equal to $2k+1$ and reverses the parity of each part in π', thus $\pi'' \in \mathcal{R}_e^{\{S_{2k+1}\}}(n)$.

(2) If π contains no parts equal to $2k+2$, map π to π'' via ψ_2 with π' as an intermediary step. First, subtract 1 from every part. The resulting partition π' reverses the parity of each part of π, and has minimum possible part size $2k$, thus $\pi' \in \mathcal{R}_e^{\{S_{2k+1}\}}(n-1)$. Now, append a new part of size $2k+2$ to π'. The resulting π'' has perimeter n and $\pi'' \in \mathcal{R}_e^{\{S_{2k+1}\}}(n)$ as we have only added an even part of size $2k$.

There exists a mapping for every $\pi \in \mathcal{R}_o^{\{S_{2k+1}\}}(n)$ since every $\pi \in \mathcal{R}_o^{\{S_{2k+1}\}}(n)$ either has a part equal to $2k+2$ or does not have a part equal to $2k+2$.

First note that fixing $a = 2k+1$, $\mathcal{R}_o^{\{S_{2k+1}\}}(n) = O_{2k+1}(n) \sqcup O^c_{2k+1}(n)$ and $\widetilde{E}(n) = \widetilde{E}_{2k+1}(n) \sqcup \widetilde{E}^c_{2k+1}(n)$. We have that φ_2 is a mapping from $O_{2k+1}(n)$ to $\widetilde{E}^c_{2k+1}(n)$ and ψ_2 is a mapping from $O^c_{2k+1}(n)$ to $\widetilde{E}_{2k+1}(n)$. Both mappings are invertible: given $\pi'' \in \widetilde{E}^c_{2k+1}(n)$, the unique corresponding $\pi \in O_{2k+1}(n)$ is found by subtracting 1 from every part in π'' and adding a new smallest part of $2k+2$. Likewise, given $\pi'' \in \widetilde{E}_{2k+1}(n)$, the unique corresponding $\pi \in O^c_{2k+1}(n)$ is found by deleting the smallest part $2k+2$ and adding one to every part.

So φ_2 and ψ_2 are bijections and thus Ψ_2 is a bijection from $\mathcal{R}_o^{\{S_{2k+1}\}}(n)$ to $\widetilde{E}(n) \subsetneq \mathcal{R}_e^{\{S_{2k+1}\}}(n)$. The strict subset follows from the fact that for any choice of n we have $\mathcal{R}_e^{\{S_{2k+1}\}}(n) \setminus \widetilde{E}(n) \neq \emptyset$. (i.e., if $n \geq 2k+4$ even then $\pi = (n) \in \mathcal{R}_e^{\{S_{2k+1}\}}(n) \setminus \widetilde{E}(n)$, and if $n \geq 2k+3$ odd then $\pi = (n-2, 2k, 2k) \in \mathcal{R}_e^{\{S_{2k+1}\}}(n) \setminus \widetilde{E}(n)$). We are left with (45).

$$r_o^{\{S_{2k+1}\}}(n) = |\mathcal{R}_o^{\{S_{2k+1}\}}(n)| = |\widetilde{E}(n)| < |\mathcal{R}_e^{\{S_{2k+1}\}}(n)| = r_e^{\{S_{2k+1}\}}(n).$$

□

We now prove a generalization of the fixed perimeter analogue of Banerjee et al.'s result (17), which is given by the special case of $\ell = 2$ below.

Theorem 5.2. *For $\ell \geq 2$ and for $n \geq \ell - 1$,*

$$r_o^{\{S_{\ell-2} \cup \ell\}}(n) < r_e^{\{S_{\ell-2} \cup \ell\}}(n) \tag{46}$$

for ℓ odd, and

$$r_e^{\{S_{\ell-2} \cup \ell\}}(n) < r_o^{\{S_{\ell-2} \cup \ell\}}(n) \tag{47}$$

for ℓ even.

Proof. The result is clear for $n = \ell - 1 \leq n \leq \ell + 1$ as the only allowed parts are $\ell - 1$ as well as $\ell + 1$ for the final case, so the following will apply to $n \geq \ell + 2$. Consider ℓ odd (resp. ℓ even) and let $\pi \in \mathcal{R}_o^{\{S_{\ell-2} \cup \ell\}}(n)$ (resp. $\pi \in \mathcal{R}_e^{\{S_{\ell-2} \cup \ell\}}(n)$). We define an injective mapping $\Psi_a = \psi_{1a} \cup \psi_{2a} \cup \psi_{3a}$ (resp. $\Psi_b = \psi_{1b} \cup \psi_{2b} \cup \psi_{3b}$) from $\mathcal{R}_o^{\{S_{\ell-2} \cup \ell\}}(n)$ to $\mathcal{R}_e^{\{S_{\ell-2} \cup \ell\}}(n)$ (resp. $\mathcal{R}_e^{\{S_{\ell-2} \cup \ell\}}(n)$ to $\mathcal{R}_o^{\{S_{\ell-2} \cup \ell\}}(n)$) as follows.

- If π contains at least one part equal to $\ell - 1$ (i.e., $\pi = (\pi_1, \ldots, \pi_{k-1}, \ell - 1)$), map π to π'' via ψ_{1a} (resp. ψ_{1b}) with π' as an intermediary step. First, delete the final part equal to $\ell - 1$. The resulting $\pi' \in \mathcal{R}_o^{\{S_{\ell-2} \cup \ell\}}(n-1)$ (resp. $\pi' \in \mathcal{R}_e^{\{S_{\ell-2} \cup \ell\}}(n-1)$) as we have only removed an even (resp. odd) part. Now, add 1 to every part in π' except any other part equal to $\ell - 1$. The resulting π'' has perimeter n and reverses the parity of every part in π except perhaps any additional parts equal to $\ell - 1$, thus $\pi'' \in \mathcal{R}_e^{\{S_{\ell-2} \cup \ell\}}(n)$ (resp. $\pi'' \in \mathcal{R}_o^{\{S_{\ell-2} \cup \ell\}}(n)$).
- If π does not contain any parts equal to $\ell - 1$ but contains $m > 1$ parts equal to $\ell + 1$, map π to π'' via ψ_{2a} (resp. ψ_{2b}) with π' as an intermediary step. First, delete 1 from every part but the part(s) equal to $\ell + 1$. Replace $m - 1$ of the parts equal to $\ell + 1$ with $m - 1$ parts equal to $\ell - 1$. The resulting partition π' reverses the parity of every part in π except perhaps any additional parts equal to $\ell + 1$, so $\pi' \in \mathcal{R}_e^{\{S_{\ell-2} \cup \ell\}}(n-1)$

(resp. $\pi' \in \mathcal{R}_o^{\{S_{\ell-2}\cup\ell\}}(n-1)$). Now, append a new smallest part of $\ell-1$. Since we have only added an even (resp. odd) part, the resulting $\pi'' \in \mathcal{R}_e^{\{S_{\ell-2}\cup\ell\}}(n)$ (resp. $\pi'' \in \mathcal{R}_o^{\{S_{\ell-2}\cup\ell\}}(n)$).

- If π does not contain any parts equal to $\ell-1$ or parts equal to $\ell+1$, then map π to π'' via ψ_{3a} (resp. ψ_{3b}) with π' as an intermediary step. First, delete 1 from every part of π. The resulting $\pi' \in \mathcal{R}_e^{\{S_{\ell-2}\cup\ell\}}(n-1)$ (resp. $\pi' \in \mathcal{R}_o^{\{S_{\ell-2}\cup\ell\}}(n-1)$). Now, we add a new smallest part of $\ell+1$. Thus, the final $\pi'' \in \mathcal{R}_e^{\{S_{\ell-2}\cup\ell\}}(n)$ (resp. $\pi' \in \mathcal{R}_o^{\{S_{\ell-2}\cup\ell\}}(n)$).

A mapping exists for every $\pi \in \mathcal{R}_o^{\{S_{\ell-2}\cup\ell\}}(n)$ (resp. $\mathcal{R}_e^{\{S_{\ell-2}\cup\ell\}}(n)$) since every π either has some part(s) equal to $\ell-1$, has no parts equal to $\ell-1$ but some part(s) equal to $\ell+1$, or has no parts equal to $\ell-1$ or equal to $\ell+1$. We introduce the following sets.

$$
\begin{aligned}
A &= \{\pi \mid \pi \text{ contains at least one part equal to } \ell-1 \text{ and at least one part equal to } \ell+1\},\\
B &= \{\pi \mid \pi \text{ contains at least one part equal to } \ell-1 \text{ and no parts equal to } \ell+1\},\\
C &= \{\pi \mid \pi \text{ contains no parts equal to } \ell-1 \text{ and at least one part equal to } \ell+1\},\\
D &= \{\pi \mid \pi \text{ contains no parts equal to } \ell-1 \text{ and no parts equal to } \ell+1\},\\
E_S(n) &= \{\pi \in \mathcal{R}_e^{\{S_{\ell-2}\cup\ell\}}(n) \cap S \mid S \in \{A,B,C,D\}\},\\
O_S(n) &= \{\pi \in \mathcal{R}_o^{\{S_{\ell-2}\cup\ell\}}(n) \cap S \mid S \in \{A,B,C,D\}\},\\
\widetilde{O}(n) &= \{\pi \in \mathcal{R}_o^{\{S_{\ell-2}\cup\ell\}}(n) \mid \omega(\pi) \geq 2\},\\
\widetilde{E}(n) &= \{\pi \in \mathcal{R}_e^{\{S_{\ell-2}\cup\ell\}}(n) \mid \omega(\pi) \leq -2\},\\
\widetilde{O}_S(n) &= \widetilde{O}(n) \cap O_S(n) \text{ and } \widetilde{E}_S(n) = \widetilde{E}(n) \cap E_S(n).
\end{aligned}
$$

Note, considering ℓ odd, that $\mathcal{R}_o^{\{S_{\ell-2}\cup\ell\}}(n) = O_A(n) \sqcup O_B(n) \sqcup O_C(n) \sqcup O_D(n)$, and $\widetilde{E}(n) = \widetilde{E}_A(n) \sqcup \widetilde{E}_B(n) \sqcup \widetilde{E}_C(n) \sqcup \widetilde{E}_D(n)$. Note ψ_{1a} is a mapping from $O_A(n) \cup O_B(n)$ to $\widetilde{E}_B(n) \cup \widetilde{E}_D(n)$, ψ_{2a} is a mapping from $O_C(n)$ to $\widetilde{E}_A(n)$, and ψ_{3a} is a mapping from $O_D(n)$ to $\widetilde{E}_C(n)$. All mappings are invertible: given $\pi'' \in \widetilde{E}_B(n) \cup \widetilde{E}_D(n)$, the unique corresponding $\pi \in O_A(n) \cup O_B(n)$ is found by deleting $\ell-1$ from every part not equal to $\ell-1$ and appending an additional smallest part of $\ell-1$. Given $\pi'' \in \widetilde{E}_A(n)$, the unique corresponding $\pi \in O_C(n)$ is found by deleting one part equal to $\ell-1$, adding one to every part but the $\ell-1$'s and one part equal to $\ell+1$, and replacing all parts equal to $\ell-1$ with parts equal to $\ell+1$. Finally, given $\pi'' \in \widetilde{E}_C(n)$, the unique corresponding $\pi \in O_D(n)$ is found by deleting one part of size $\ell+1$ and adding $\ell-1$ to every remaining part.

So ψ_{1a}, ψ_{2a} and ψ_{3a} are bijections and thus Ψ_a is a bijection from $\mathcal{R}_o^{\{S_{\ell-2}\cup\ell\}}(n)$ to $\widetilde{E}(n) \subsetneq \mathcal{R}_e^{\{S_{\ell-2}\cup\ell\}}(n)$. The strict subset comes from the fact that for any choice of n we have $\mathcal{R}_e^{\{S_{\ell-2}\cup\ell\}}(n) \setminus \widetilde{E}(n) \neq \emptyset$. If $n \geq \ell+2$ even, then $\pi = (n) \in \mathcal{R}_e^{\{S_{\ell-2}\cup\ell\}}(n) \setminus \widetilde{E}(n)$, and if $n \geq \ell+3$, then $\pi = (n, n-1, n-1) \in \mathcal{R}_e^{\{S_{\ell-2}\cup\ell\}}(n) \setminus \widetilde{E}(n)$. We are left with (46).

$$
r_o^{\{S_{\ell-2}\cup\ell\}}(n) = \mid \mathcal{R}_o^{\{S_{\ell-2}\cup\ell\}}(n)| = |\widetilde{E}(n)| < |\mathcal{R}_e^{\{S_{\ell-2}\cup\ell\}}(n)| = r_e^{\{S_{\ell-2}\cup\ell\}}(n).
$$

Likewise, for ℓ even, $\mathcal{R}_e^{\{S_{\ell-2}\cup\ell\}}(n) = E_A(n) \sqcup E_B(n) \sqcup E_C(n) \sqcup E_D(n)$ and $\widetilde{O}(n) = \widetilde{O}_A(n) \sqcup \widetilde{O}_B(n) \sqcup \widetilde{O}_C(n) \sqcup \widetilde{O}_D(n)$. We may observe that ψ_{1b} is a mapping from $E_A(n) \cup E_B(n)$ to $\widetilde{O}_B(n) \cup \widetilde{O}_D(n)$, ψ_{2b} is a mapping from $E_C(n)$ to $\widetilde{O}_A(n)$, and $\psi_{3b}(n)$ is a mapping from $E_D(n)$ to $\widetilde{O}_C(n)$.

All mappings are invertible by the same process as the analogous mappings for $\ell - 1$ odd, so ψ_{1b}, ψ_{2b} and ψ_{3b} are bijections and thus Ψ_b is a bijection from $\mathcal{R}_e^{\{S_{\ell-2}\cup\ell\}}(n)$ to $\widetilde{O}(n) \subsetneq \mathcal{R}_o^{\{S_{\ell-2}\cup\ell\}}(n)$. The strict subset comes from the fact that for any choice of n we have $\mathcal{R}_o^{\{S_{\ell-2}\cup\ell\}}(n) \setminus \widetilde{O}(n) \neq \emptyset$. If $n \geq \ell+3$ odd then $\pi = (n) \in \mathcal{R}_o^{\{S_{\ell-2}\cup\ell\}}(n) \setminus \widetilde{O}(n))$, and if $n \geq \ell+2$ even then $\pi = (n-2, n-1, n-1) \in \mathcal{R}_o^{\{S_{\ell-2}\cup\ell\}}(n) \setminus \widetilde{O}(n))$. We are left with (47).

$$r_e^{\{S_{\ell-2}\cup\ell\}}(n) = |\mathcal{R}_e^{\{S_{\ell-2}\cup\ell\}}(n)| = |\widetilde{O}(n)| < |\mathcal{R}_o^{\{S_{\ell-2}\cup\ell\}}(n)| = r_o^{\{S_{\ell-2}\cup\ell\}}(n).$$

□

5.2. **Parity Bias for Partitions with Fixed Degree of Bias.** Let $r(m,n)$ denote the number of perimeter n partitions π such that $\omega(\pi) = m$. We may consider fixed degree of bias as a method of decomposing fixed perimeter partitions. In particular, we have

$$r(n) = r(0,n) + r_e(n) + r_o(n) = r(0,n) + \sum_{m=1}^{n} \big(r(m,n) + r(-m,n)\big).$$

We find that the direction of inequality in our fixed perimeter analogue of (19) depends on the parity of n, and is obtained by taking $m = 1$ below. We are also able to generalize this result to any choice of m. Namely,

Theorem 5.3. *Let $n \geq m \geq 1$. For m, n of the same parity,*

$$r(m,n) > r(-m,n). \tag{48}$$

For m, n of opposite parity,

$$r(-m,n) > r(m,n). \tag{49}$$

Proof. The result is clear for $n = m$, $m+1$. The following argument applies for $n \geq m+2$. We first consider m, n both odd. Note that a partition in $\mathcal{R}(m,n)$ or $\mathcal{R}(-m,n)$ must have an odd number of parts λ in order to have a bias of m odd, and thus must have largest part α odd to have $n = \alpha + \lambda - 1$ odd. We claim that for any choice of α, there are more partitions in $\mathcal{R}(m,n)$ than in $\mathcal{R}(-m,n)$.

For a fixed α, let S_o (resp. S_e) denote the set of odd parts (resp. even parts) less than or equal to α. Using a stars and bars counting technique to select both the remaining odd parts after fixing α and the even parts, we have that the number of ways to choose exactly m more even parts than odd parts (including α in the total bias count) is given by

$$\binom{|S_o| + \frac{\lambda-m-2}{2} - 1}{\frac{\lambda-m-2}{2}}\binom{|S_e| + \frac{\lambda+m}{2} - 1}{\frac{\lambda+m}{2}}.$$

Likewise, the number of ways to choose exactly m more odd parts (including α in the total bias count) than even parts is given by

$$\binom{|S_o| + \frac{\lambda+m-2}{2} - 1}{\frac{\lambda+m-2}{2}}\binom{|S_e| + \frac{\lambda-m}{2} - 1}{\frac{\lambda-m}{2}}.$$

Suppose we have fixed $\alpha = 2k + 1$. Then $|S_o| = k + 1, |S_e| = k$, and we may rewrite $\lambda = n - \alpha + 1 = n - 2k$. Substituting in and summing over all possible choices of k yielding valid choices of α, we have for a given n that

$$r(-m, n) = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{k + 1 + \frac{n-2k-m-2}{2} - 1}{\frac{n-2k-m-2}{2}}\binom{k + \frac{n-2k+m}{2} - 1}{\frac{n-2k+m}{2}}, \tag{50}$$

$$r(m, n) = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{k + 1 + \frac{n-2k+m-2}{2} - 1}{\frac{n-2k+m-2}{2}}\binom{k + \frac{n-2k-m}{2} - 1}{\frac{n-2k-m}{2}}. \tag{51}$$

We claim each term in the sum in (51) is greater than the corresponding term in (50). Namely,

$$\begin{aligned}
&\binom{k + 1 + \frac{n-2k-m-2}{2} - 1}{\frac{n-2k-m-2}{2}}\binom{k + \frac{n-2k+m}{2} - 1}{\frac{n-2k+m}{2}} = \binom{\frac{n-m-2}{2}}{\frac{n-m-2}{2} - k}\binom{\frac{n+m-2}{2}}{\frac{n+m}{2} - k} \\
&= \frac{\left(\frac{n-m-2}{2} \cdot \ldots \cdot \left(\frac{n-m-2}{2} - k + 1\right)\right) \cdot \left(\frac{n-m-2}{2} + m\right) \cdot \ldots \cdot \left(\frac{n-m-2}{2} + m - k + 2\right)\Big)}{k!(k-1)!} \\
&< \frac{\left(\frac{n-m-2}{2} \cdot \ldots \cdot \left(\frac{n-m-2}{2} - k + 2\right)\right) \cdot \left(\left(\frac{n-m-2}{2} + m\right) \cdot \ldots \cdot \left(\frac{n-m-2}{2} + m - k + 1\right)\right)}{k!(k-1)!} \\
&= \binom{\frac{n+m-2}{2}}{\frac{n+m-2}{2} - k}\binom{\frac{n-m-2}{2}}{\frac{n-m}{2} - k} = \binom{k + 1 + \frac{n-2k+m-2}{2} - 1}{\frac{n-2k+m-2}{2}}\binom{k + \frac{n-2k-m}{2} - 1}{\frac{n-2k-m}{2}},
\end{aligned}$$

where the inequality follows from the fact that $(\frac{n-m-2}{2} - k + 1) < (\frac{n-m-2}{2} + m - k + 1)$. So (48) holds for n, m both odd. For n, m both even, we have $\alpha = 2k + 1$ odd and $\lambda = n - 2k$ even. Then, given n, $r(-m, n)$ is given by (50) and $r(m, n)$ is given by (51). So (48) holds for n, m both even by the same reasoning.

For n even, m odd, we still must have an odd number of parts λ to have m odd, thus we have $\alpha = 2k$ even so that $n - \alpha - 1$ is even. So, $|S_o| = |S_e| = k$ and $\lambda = n - 2k + 1$. Then, for a given n,

$$r(m, n) = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{k + \frac{n-2k+m+1}{2} - 1}{\frac{n-2k+m+1}{2}}\binom{k + \frac{n-2k-m-1}{2} - 1}{\frac{n-2k-m-1}{2}}, \tag{52}$$

$$r(-m, n) = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{k + \frac{n-2k-m+1}{2} - 1}{\frac{n-2k-m+1}{2}}\binom{k + \frac{n-2k+m-1}{2} - 1}{\frac{n-2k+m-1}{2}}. \tag{53}$$

Each term in the sum in (53) is larger than the corresponding term in (52). In particular,

$$\binom{k+\frac{n-2k+m+1}{2}-1}{\frac{n-2k+m+1}{2}}\binom{k+\frac{n-2k-m-1}{2}-1}{\frac{n-2k-m-1}{2}}=\binom{\frac{n+m-1}{2}}{\frac{n+m+1}{2}-k}\binom{\frac{n-m-3}{2}}{\frac{n-m-1}{2}-k}$$
$$=\frac{\left((\frac{n-m-1}{2}+m)\cdot\ldots\cdot(\frac{n-m-1}{2}+m-k+2)\right)\cdot\left((\frac{n-m-1}{2}-1)\cdot\ldots\cdot(\frac{n-m-1}{2}-k+1)\right)}{((k-1)!)^2}$$
$$<\frac{\left((\frac{n-m-1}{2}+m-1)\cdot\ldots\cdot(\frac{n-m-1}{2}+m-k+1)\right)\cdot\left(\frac{n-m-1}{2}\cdot\ldots\cdot(\frac{n-m-1}{2}-k+2)\right)}{((k-1)!)^2}$$
$$=\binom{\frac{n-m-1}{2}}{\frac{n-m+1}{2}-k}\binom{\frac{n+m-3}{2}}{\frac{n+m-1}{2}-k}=\binom{k+\frac{n-2k-m+1}{2}-1}{\frac{n-2k-m+1}{2}}\binom{k+\frac{n-2k+m-1}{2}-1}{\frac{n-2k+m-1}{2}},$$

where the inequality follows from the fact that $(\frac{n-m-1}{2}+m)\cdot(\frac{n-m-1}{2}-k+1)<(\frac{n-m-1}{2}+m-k+1)\cdot(\frac{n-m-1}{2})$. So (49) holds for n even, m odd.

For n odd, m even, we now have $\alpha=2k$ even, so $|S_o|=|S_e|=k$, and $\lambda=n-2k+1$ even. Then, given n, $r(m,n)$ is given by (52), and $r(m,n)$ is given by (53). Then (49) holds for the same reasoning as above. □

5.3. **Other Bias Generalizations.** Let $rd_o(n)$ (resp. $rd_e(n)$) denote the number of perimeter n partitions π with distinct parts and $\omega(\pi)\geq 1$ (resp. $\omega(\pi)\leq -1$). We conjecture the following fixed perimeter analogue of Theorem 2.4.

Conjecture 5.4. *For all n odd and $n\geq 10$ even,*

$$rd_o(n)>rd_e(n).$$

Computational evidence suggests the above inequality holds, and we prove the following lemma that may be helpful in future research.

Lemma 5.5. *For all n,*

$$rd_o(n)=\sum_{k=1}^{\lfloor\frac{n}{2}\rfloor}\sum_{a=\lceil\frac{n-2k}{2}\rceil+1}^{n-2k}\binom{k}{a}\binom{k-1}{n-2k-a}+\sum_{k=1}^{\lfloor\frac{n}{2}\rfloor}\sum_{a=\lceil\frac{n-2k-1}{2}\rceil}^{n-2k-1}\binom{k}{a}\binom{k}{n-2k-1-a},\tag{54}$$

(55)

$$rd_e(n)=\sum_{k=1}^{\lfloor\frac{n}{2}\rfloor}\sum_{a=\lceil\frac{n-2k}{2}\rceil}^{n-2k}\binom{k-1}{a}\binom{k}{n-2k-a}+\sum_{k=1}^{\lfloor\frac{n}{2}\rfloor}\sum_{a=\lceil\frac{n-2k-1}{2}\rceil+1}^{n-2k-1}\binom{k}{a}\binom{k}{n-2k-1-a}.$$

Proof. The first term in each represents choices of $\alpha=2k$ even, and the second $\alpha=2k+1$ odd. The index k then sums over all possible choices of α, and the index a tracks the number of odd parts chosen in (54) and the number of even parts chosen in (55) out of the remaining $n-\alpha$ parts to be chosen. For fixed a and fixed α, there then remain $n-2k-a$ parts to choose for α even or $n-2k-1-a$ for α odd. For $\alpha=2k$ even, there are k choices of distinct odds and $k-1$ choices of distinct evens smaller than α' for $\alpha=2k+1$ odd, there are k distinct odds and k distinct evens smaller than α.

For $\alpha=2k$ even, in order to have more odd parts than even parts overall, we must choose at least $\lceil\frac{n-2k}{2}\rceil+1$ odd parts; in order to have more even parts than odd parts, we must choose at least $\lceil\frac{n-2k}{2}\rceil$ even parts in addition to α. Likewise, for $\alpha=2k+1$ odd, in order to have more odd parts than even parts overall, we must choose at least $\lceil \frac{n-2k}{2} \rceil$ odd parts in addition to α; and, in order to have more even parts than odd parts, we must choose at least $\lceil \frac{n-2k}{2} \rceil + 1$ even parts. □

Although we haven't yet proved Conjecture 5.4, we can prove the fixed perimeter analogue of Theorem 2.5. Let $r_{a,b,m}(n)$ and $r_{b,a,m}(n)$ denote the number of partitions of perimeter n with more parts congruent to $a \pmod m$ than parts congruent to $b \pmod m$ (resp. more parts congruent to $b \pmod m$ than parts congruent to $a \pmod m$). We have the following.

Theorem 5.6. *For all $n \geqslant 1$ and $1 \leqslant a < b \leqslant m$,*

$$r_{a,b,m}(n) \geqslant r_{b,a,m}(n).$$

Note that Theorem 1.7 gives the case of Theorem 5.6 given by $r_{1,2,2}(n) \geq r_{2,1,2}(n)$ (allowing for $n = 2$ by dropping the requirement that the inequality be strict).

Proof. For fixed a, b, both quantities are 0 when $n < a$ and the theorem holds with equality. The following mapping considers only choices of a, n such that $n \geq a$. Consider $\pi \in \mathcal{R}_{b,a,m}(n)$. Define $D_1 = b - a$ and $D_2 = a + m - b$ to denote the distances between residue classes. For a given π_1, let $B < \pi_1$ denote the nearest number to π_1 congruent to $b \pmod m$ and $A < \pi_1$ denote the nearest number to π_1 congruent to $b \pmod m$. We now define an injective mapping $\Psi : \mathcal{R}_{b,a,m}(n) \to \mathcal{R}_{a,b,m}(n)$.

Case 1: π contains no parts $\equiv a \pmod m$

(1) The partition π must contain some number of parts $\equiv b \pmod m$. As $b > a$, it is clear from Theorem 1.5 that $r(n \mid \text{all parts } \equiv b \pmod m) \leq r(n \mid \text{all parts } \equiv a \pmod m)$. Then the image of partitions containing only parts So we need only consider partitions having at least one part $\not\equiv a, b \pmod m$. Map π to π'' via ψ_1 as follows: replace each such part π_b with a corresponding part of size $\pi_b - D_1$. Reorder the parts as needed to maintain non-increasing order. The resulting partition π' has perimeter $n - D_1 \leq n' \leq n$. Further, π' is clearly unique to π and no two π with either distinct configurations of parts $\equiv a \pmod m$ or distinct configurations of parts $\not\equiv a \pmod m$ will map to the same π'. Otherwise, this would result in distinct configurations of parts $\equiv b \pmod m$ or distinct configurations of parts $\not\equiv b \pmod m$, respectively. Thus the mapping from π to π' is an injection.

Let $s < m$ be the smallest positive integer $s \neq a$, b. Such an s exists for every legal choice of a, b, m except for $a = 1$, $b = 2$, $m = 2$, for which is given by Theorem 1.7. Generate π'' from π by adding $n - n'$ parts of size s to π' and reordering the parts as needed to maintain non-increasing order. Then $\pi'' \in \mathcal{R}_{a,b,m}(n)$. The value of s is recoverable from π'' and thus the unique π' is identifiable, so the mapping from π' to π'' is also an injection, thus ψ_1 is an injection.

Case 2: π contains at least one part $\equiv a \pmod m$.

(1) If π_1 is not congruent to a or $b \pmod m$ and $B > A$, map π to π' via ψ_2. First replace each part π_b congruent to $b \pmod m$ with a corresponding part of size $\pi_b - D_1$ and each part π_a congruent to $a \pmod m$ with a corresponding part of size $\pi_a - D_2$ if $\pi_a > a$ or with a part of size B if $\pi_a = a$. Reorder the parts as needed to maintain non-increasing order, noting that π_1 will not be reordered and perimeter remains

constant. The residue class of each π_a and π_b has switched, thus $\pi' \in \mathcal{R}_{a,b,m}(n)$. Note that π is uniquely recoverable from π' by replacing parts π'_a and $\pi'_b \neq B$ with parts equal to $\pi'_a + D_1$ and $\pi'_b + D_2$, respectively, and by replacing parts of size B with parts of size a; thus ψ_2 is an injection.

(2) If π_1 is not congruent to a or b (mod m) and $A > B$, map π to π' via ψ_3 as follows: replace each part π_b congruent to b (mod m) with a corresponding part of size $\pi_b - D_1$. Replace each part π_a congruent to a (mod m) except for any parts of size A with a corresponding part of size $\pi_a - D_2$ if $\pi_a > a$ and with a part of size B if $\pi_a = a$. Reorder the part as needed to maintain non-increasing order, noting that π_1 will not be reordered and perimeter remains constant. The residue class of each π_a and π_b has switched except for those parts of size A, thus $\pi' \in \mathcal{R}_{a,b,m}(n)$. Note that π is uniquely recoverable from π' by replacing parts $\pi'_a \neq A$ and π'_b with parts equal to $\pi'_a + D_1$ and $\pi'_b + D_2$, respectively; thus ψ_3 is an injection.

(3) If $\pi_1 \equiv a$ (mod m) or $\pi_1 \equiv b$ (mod m) and condition $(*)$ below does not apply, map π to π' via ψ_4. Replace each part π_b congruent to b (mod m) with a corresponding part of size $\pi_b - D_1$ if $\pi_b > b$. If $\pi_b = b$, replace π_b with a part of size π_1 if $\pi_1 \equiv a$ or with a part of size A if $\pi_1 \equiv b$. Replace each part π_a congruent to a (mod m) with a corresponding part of size $\pi_a - D_2$ if $\pi_a > a$. If $\pi_a = a$, replace π_a with a part of size π_1 if $\pi_1 \equiv b$ or with a part of size B if $\pi_1 \equiv a$. Reorder the parts as needed to maintain non-increasing order. Repeat this process as many times as needed (with a total odd number of repeats) to get back to a largest part of π_1. The existence of such a point after a total odd number of repeats is guaranteed by the exclusion of partitions satisfying $(*)$. The final perimeter is still n and the residue class of each π_a and π_b has switched, thus $\pi' \in \mathcal{R}_{a,b,m}(n)$. Note that π is uniquely recoverable from π' by reversing the process the minimum odd number of times needed to get back to the original π_1; thus ψ_4 is an injection.

(4) Condition $(*)$: Either $\pi_1 \equiv a$ and every part $\pi_i \equiv b$ is of the form $b + t_i m$ with every t_i odd or $\pi_1 \equiv b$ and every part $\pi_i \equiv a$ is of the form $b + t_i m$ with every t odd. In this case, we will never reach a part of size a or b on an even number of repeats, so we adjust the process described in case (3) to return to a largest part of π_1 with an odd number of repeats to preserve the desired direction of bias. Define ψ'_4 to be the same as ψ_4 except that now we discard the conditions concerning $\pi_a = a$ and $\pi_b = b$ and replace them: if $\pi_b = b + m$, replace π_b with a part of size π_1 if $\pi_1 \equiv a$ or with a part of size A if $\pi_1 \equiv b$. If $\pi_a = a + m$, replace π_a with a part of size π_1 if $\pi_1 \equiv b$ or with a part of size B if $\pi_1 \equiv a$. Note that π is uniquely recoverable by the same reasoning as case (3) and thus ψ'_4 is an injection.

A given $\pi \in \mathcal{R}_{a,b,m}(n)$ with $n \geq a$ must satisfy exactly one of the conditions of having no parts $\equiv a$ (mod m) or having at least one part $\equiv a$ (mod m). Given the latter case, π satisfies exactly one of π_1 not congruent to a or b (mod m) and $B > A$, π_1 not congruent to a or b (mod m) and $A < B$, or $\pi_1 \equiv a$ or b (mod m) with $(*)$ either applying or not applying. Thus exactly one mapping ψ_i applies to a given π.

We further note that the images of each ψ_i are disjoint. In particular, partitions $\pi \in \mathcal{R}_{b,a,m}(n)$ containing no parts $\equiv a$ (mod m) and at least one part $\not\equiv a, b$ (mod m) have their image under ψ_1 in the set of partitions $\pi' \in \mathcal{R}_{a,b,m}(n)$ containing no parts $\equiv b$ (mod m) and at least one part $\not\equiv a, b \pmod{m}$. Partitions $\pi \in \mathcal{R}_{b,a,m}(n)$ having at least one part $\equiv a$ (mod m and largest part $\not\equiv a, b \pmod{m}$ with $B > A$ (resp. $A > B$) have their image under ψ_2 (resp. ψ_3) in the set of partitions $\pi' \in \mathcal{R}_{a,b,m}(n)$ having at least one part $\equiv b$ (mod m and largest part $\not\equiv a, b \pmod{m}$ with $B > A$ (resp. $A > B$). Partitions $\pi \in \mathcal{R}_{b,a,m}(n)$ with $\pi_1 \equiv a$ or $\pi_1 \equiv b \pmod{m}$ not satisfying condition $(*)$ (resp. satisfying condition $(*)$) have their image under ψ_4 (resp. ψ_4') in the set of partitions $\pi \in \mathcal{R}_{b,a,m}(n)$ not satisfying condition $(*)$ (resp. satisfying condition $(*)$).

Then since each ψ_i is injective and their images are disjoint, $\Psi = \psi_1 \cup \psi_2 \cup \psi_3 \cup \psi_4 \cup \psi_4'$ is injective. $\square$

5.4. **Analogues of PED and POD Partitions.** Another lens through which we may consider parity in partitions is through PED and POD partitions. Let $p_{ed}(n)$ (resp. $p_{od}(n)$) denote the number of partitions of n where all even parts (resp. all odd parts) must be distinct but odd parts (resp. even parts) have no restrictions. There have been a number of developments in the occurrence and arithmetic properties of $p_{ed}(n)$ and $p_{od}(n)$, for instance see [4, 6, 7, 9, 10, 24]. We use a combinatorial approach to obtain an exact count of these partitions in the fixed perimeter setting, which we will denote by $r_{ed}(n)$ and $r_{od}(n)$. We further find that we can apply the injections developed in Section 5.1 to obtain similar results to fixed perimeter parity bias in the context of fixed perimeter PED and POD partitions.

Theorem 5.7. *For all $n \geq 1$,*

$$r_{ed}(n) = \sum_{\alpha=1}^{n} \sum_{b=0}^{n-\alpha} \binom{\lfloor \frac{\alpha-1}{2} \rfloor}{b} \binom{\lceil \frac{\alpha}{2} \rceil + n - \alpha - b - 1}{n - \alpha - b}$$

and

$$r_{od}(n) = \sum_{\alpha=1}^{n} \sum_{b=0}^{n-\alpha} \binom{\lceil \frac{\alpha-1}{2} \rceil}{b} \binom{\lfloor \frac{\alpha}{2} \rfloor + n - \alpha - b - 1}{n - \alpha - b}.$$

For all $n \neq 1, 3$,

$$r_{ed}(n) > r_{od}(n). \tag{56}$$

Proof. The index α tracks the size of the largest part, and the index b tracks the number of distinct even (resp. distinct odd) parts smaller than α that are chosen. Once α is fixed, we choose the remaining $n - \alpha$ parts by first selecting the (possibly zero) distinct even (resp. distinct odd) parts, and then selecting the (possibly zero) remaining $n - \alpha - b$ unrestricted odd (resp. unrestricted even) parts.

There are $\lfloor \frac{\alpha-1}{2} \rfloor$ choices of distinct evens smaller than α and $\lceil \frac{\alpha-1}{2} \rceil$ choices of distinct odds smaller than α, giving us the first binomial coefficient in each equation. Likewise, there are $\lceil \frac{\alpha}{2} \rceil$ odds smaller than or equal to α and $\lfloor \frac{\alpha}{2} \rfloor$ evens. Using a stars and bars counting method, we obtain the second binomial coefficient in each equation to count the parts not required to be distinct. The desired inequality now follows by taking $k = 0$ in Theorem 5.10. $\square$

We also find that the terms in the expansions of $r_{ed}(n)$ and $r_{od}(n)$ given by Theorem 5.7 are each enumerated recursively by a Tribonacci pattern, which we shall prove using comparison of generating functions. Such a recursion comes as an interesting contrast to

Straub's finding [26] that the number of partitions of perimeter n into distinct parts is given by the Fibonacci number F_n. To show this we first require the following lemma.

Lemma 5.8. *The tribonacci sequence $\{T_n\}_{n=1}^{\infty}$ with initial values T_1, T_2, T_3 has the generating function*

$$\sum_{n=1}^{\infty} T_n q^n = \frac{T_1 q + (T_2 - T_1)q^2 + (T_3 - T_2 - T_1)q^3}{1 - q - q^2 - q^3}.$$

Proof. We begin by noting that $\{T_n\}_{n=1}^{\infty}$ is defined as $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ with initial values T_1, T_2, T_3. Now, consider the following.

$$\begin{aligned}(1-q-q^2-q^3)\sum_{n=1}^{\infty} T_n q^n &= T_1 q + (T_2 - T_1)q^2 + (T_3 - T_2 - T_1)q^3 + \sum_{n=4}^{\infty}(T_n - T_{n-1} - T_{n-2} - T_{n-3}) \\ &= T_1 q + (T_2 - T_1)q^2 + (T_3 - T_2 - T_1)q^3.\end{aligned}$$

Thus,

$$\sum_{n=1}^{\infty} T_n q^n = \frac{T_1 q + (T_2 - T_1)q^2 + (T_3 - T_2 - T_1)q^3}{1 - q - q^2 - q^3}.$$

□

We shall now prove a recursion for $r_{ed}(n)$ and $r_{od}(n)$ using the above lemma.

Theorem 5.9. *For $n \geq 1$,*

$$\begin{aligned} r_{ed}(n+3) &= r_{ed}(n) + r_{ed}(n+1) + r_{ed}(n+2), \\ r_{od}(n+3) &= r_{od}(n) + r_{od}(n+1) + r_{od}(n+2). \end{aligned}$$

Furthermore, $r_{ed}(n)$ is enumerated by the Tribonacci sequence $\{T_n\}_{n=1}^{\infty}$ with initial values $T_1 = 1$, $T_2 = 2$, $T_3 = 3$. Similarly, $r_{od}(n)$ is enumerated by the Tribonacci sequence $\{T'_n\}_{n=1}^{\infty}$ with initial values $T'_1 = 1$, $T'_2 = 1$, $T'_3 = 3$.

Proof. We shall first derive generating functions for $r_{ed}(n)$ and $r_{od}(n)$, beginning with $r_{ed}(n)$. We shall construct our generating functions using the profile of a partition, a process which has been described in Section 4.

As we are required to have even parts occuring at most once, we shall use odd parts as our baseline. Between two odd part sizes, we can either move ENE to include a distinct even part, or more EE to skip to the next odd part size. At a given odd part, we also allow for a move N, indicating an inclusion of such a part. So, beginning at an odd part, we must choose from N, EE, ENE. After any of these are chosen, our part size remains odd, so we repeat this process. This can occur j times for $j \in \mathbb{N}_0$, giving us the generating function block

$$1 + (yq + x^2q^2 + x^2yq^3) + (yq + x^2q^2 + x^2yq^3)^2 + \cdots = \frac{1}{1 - (yq + x^2q^2 + x^2yq^3)}.$$

Now, we must consider our initial and final moves. We begin with a single move E as required by the profile. Since a single move E brings us to a part of size 1, which is odd, we can then move to the process described above. This is given by the form (xq).

To finish, we can either end on an odd part or end on a distinct even part. From a given odd part size in the body, these take the form N and EN respectively. This is given by the form $(y + xyq)$. Note that we omit a q from both of the final options, ensuring that $n = \alpha + \lambda - 1$. Now, putting all of this together, we are left with the form

$$\sum_{\alpha=0}^{\infty}\sum_{\lambda=0}^{\infty} r_{ed}(\alpha, \lambda, n)x^{\alpha}y^{\lambda}q^{n} = \frac{xq(y + xyq)}{1 - (yq + x^2q^2 + x^2yq^3)}.$$

Now, by taking $x = y = 1$, we have that

$$\sum_{n=1}^{\infty} r_{ed}(n)q^n = \frac{q + q^2}{1 - q - q^2 - q^3}.$$

By Lemma 5.8, it is evident that this is also the generating function for the Tribonacci sequence $\{T_n\}_{n=1}^{\infty}$ with $T_1 = 1, T_2 = 2, T_3 = 3$. Thus, $r_{ed}(n) = T_n$, and in particular we have that

$$r_{ed}(n + 3) = r_{ed}(n) + r_{ed}(n + 1) + r_{ed}(n + 2).$$

Now, deriving the generating function for $r_{od}(n)$ is similar, except that we shall use even parts as our baseline. From a given even part size, we once again must choose from N, EE, ENE, giving us the body of

$$\frac{1}{1 - (yq + x^2q^2 + x^2yq^3)}.$$

Here, however, our initial and final moves become more complicated. After our required move E our part is odd, and we would like to move to an even part to begin the process described above. To do so, we must either take a step E or include a distinct odd part of size 1 given by NE. Together, our initial moves take the form $(x^2q^2 + x^2yq^2)$.

To finish, we once again can end on the current even part or end on the distinct next odd part, giving us N and EN respectively. This is given by the form $(y + xyq)$, omitting a q as done previously. Together, we have

$$\frac{(x^2q^2 + x^2yq^2)(y + xyq)}{1 - (yq + x^2q^2 + x^2yq^3)}.$$

This function, however, does not include the single partition of perimeter 1 consisting of xyq, so we must append this. Our final function becomes

$$\begin{aligned}\sum_{\alpha=0}^{\infty}\sum_{\lambda=0}^{\infty} r_{od}(\alpha, \lambda, n)x^{\alpha}y^{\lambda}q^{n} =& \frac{(x^2q^2 + x^2yq^2)(y + xyq)}{1 - (yq + x^2q^2 + x^2yq^3)} + xyq \\ =& \frac{xyq(1 + (x - y)q + xyq^2)}{1 - (yq + x^2q^2 + x^2yq^3)}.\end{aligned}$$

After taking $x = y = 1$, we have

$$\sum_{n=1}^{\infty} r_{od}(n)q^n = \frac{q+q^3}{1-q-q^2-q^3}.$$

Once again by Lemma 5.8, it is evident that this is the generating function for the Tribonacci sequence $\{T'_n\}_{n=1}^{\infty}$ with $T'_1 = 1, T'_2 = 1, T'_3 = 3$. Thus, $r_{od}(n) = T'_n$, and in particular we have that

$$r_{od}(n+3) = r_{od}(n) + r_{od}(n+1) + r_{od}(n+2).$$

□

Theorem 5.9 gives us further intuition for the result of (56) in Theorem 5.7, as we have that

$$r_{ed}(2) = T_2 > T'_2 = r_{od}(2),$$

which implies that for $n = 2$ and all $n \geq 4$, it must be that $r_{ed}(n) > r_{od}(n)$. We may now consider, as in Section 5.1, how this inequality is impacted when we impose either restriction of allowed parts or restriction of smallest part. An alternative proof of Theorem 5.7 is given by taking $k = 0$ in the first inequality below.

Theorem 5.10. *Let $k \in \mathbb{N}_0$. For all $n = 2k+2$ and $n \geq 2k+4$,*

$$r_{ed}^{\{S_{2k}\}}(n) > r_{od}^{\{S_{2k}\}}(n),$$

and for all $n = 2k+3$ and $n \geq 2k+5$,

$$r_{ed}^{\{S_{2k+1}\}}(n) < r_{od}^{\{S_{2k+1}\}}(n).$$

Proof. The result is clear for the two partitions allowed each by the restrictions when $n = 2k+2$ and when $n = 2k+3$, so we need only consider $n \geq 2k+4$ for the first inequality and $n \geq 2k+3$ for the second. The result follows from applying injections Ψ_1 and Ψ_2, respectively, from Theorem 5.1. We remark that in this context, the strict inequality follows from the fact that partitions $\pi \in \mathcal{R}_{ed}^{\{S_{2k}\}}(n)$ (resp. $\pi \in \mathcal{R}_{od}^{\{S_{2k+1}\}}(n)$) having smallest part of size $2k+2$ (resp. $2k+3$) are not in the image of $\mathcal{R}_{od}^{\{S_{2k}\}}(n)$ under Ψ_1 (resp. the image of $\mathcal{R}_{ed}^{\{S_{2k+1}\}}(n)$ under Ψ_2). □

Similarly, we have the following analogue of Theorem 5.2.

Theorem 5.11. *For $\ell \geq 2$ and for all $n \geq \ell$,*

$$r_{ed}^{\{S_{\ell-2} \cup \ell\}}(n) < r_{od}^{\{S_{\ell-2} \cup \ell\}}(n)$$

for ℓ odd, and

$$r_{od}^{\{S_{\ell-2} \cup \ell\}}(n) < r_{ed}^{\{S_{\ell-2} \cup \ell\}}(n)$$

for ℓ even.

Proof. The result is clear for $n = \ell, \ell+1$ and follows for $n \geq \ell+2$ from applying injections Ψ_a and Ψ_b, respectively, from Theorem 5.2. In this context, the strict inequality follows from the fact that partitions $\pi \in \mathcal{R}_{od}^{\{S_{\ell-2} \cup \ell\}}(n)$ (resp. $\pi \in \mathcal{R}_{ed}^{\{S_{\ell-2} \cup \ell\}}(n)$) having a part of size $\ell - 1$ occurring two or more times are not in the image of $\mathcal{R}_{ed}^{\{S_{\ell-2} \cup \ell\}}(n)$ under Ψ_a (resp. the image of $\mathcal{R}_{od}^{\{S_{\ell-2} \cup \ell\}}(n)$ under Ψ_b). □

For $1 \leq a < b \leq m$, let $r_{pam}(n)$ and $r_{pbm}(n)$, respectively, denote the number of partitions of perimeter n such that any parts $\equiv a \pmod{m}$ (resp. $\equiv b \pmod{m}$) must be distinct and all other parts are unrestricted. We may observe that $r_{od}(n)$ and $r_{ed}(n)$ are instances of $r_{pam}(n)$ and $r_{pbm}(n)$ given by $a = 1, b = 2, m = 2$. We now present the following analogue of Theorem 5.6.

Theorem 5.12. *For $n \geq 1$,*

$$r_{pbm}(n) \geq r_{pam}(n).$$

Proof. Observe that partitions containing no parts congruent to either a or b belong in both $\mathcal{R}_{pbm}(n)$ and $\mathcal{R}_{pam}(n)$, thus we may assume for comparison purposes that any partition we consider counted by either $r_{pam}(n)$ or $r_{pbm}(n)$ has at least one part congruent to either a or b. The result then follows from applying the injection Ψ from Theorem 5.6. □

6. Concluding Remarks

It is interesting to continue the exploration of analogues of regular partition results in the fixed perimeter context, a process which we have begun in this paper. In particular, it would be interesting to address Conjecture 4.1 in determining the existence of an N for with $FD_{j,k}(n) \geq FO_{j,k}(n)$ when $n \geq N$.

We also think it would be meaningful to further extend the fixed perimeter analogue of Andrews' S-T Theorem given in Theorem 1.5 by loosening the requirement of inequality in the smaller elements of S and T, as stated in Conjecture 4.8.

In Section 4.4 and Conjecture 4.9, we detail initial steps in proving an analogue of Kang and Kim's result, stated in Theorem 2.2, in the fixed perimeter setting.

In Section 5.3 and Conjecture 5.4, we also provide initial steps in proving a fixed perimeter analogue of Banerjee et al.'s parity bias result for partitions into distinct parts, given in Theorem 2.4. Both of the previous two conjectures appear to hold from computational evidence, and it would be interesting to complete the proofs of each.

Finally, it could be illuminating to develop a combinatorial proof of Lemma 3.3 in the style of Ballantine and Welch [8, Thm. 4].

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