OSU Department of Mathematics
Qualifying Examination
Spring 2024

Real Analysis

Instructions:

- Do **any three** of the four problems.
- Use separate sheets of paper for each problem. Clearly indicate the problem and page number (if several pages are used for a solution) on the top of the page.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have **four** hours to complete this examination.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination:
  1. Use the problem selection sheet to indicate your identification number and the three problems which you wish to be graded.
  2. Arrange your solutions according to the problem order with the problem selection sheet on top and any scratch-work on the bottom.
  3. Submit the exam: place your solutions together with the selection sheet and scratch paper, in the order arranged as above, into the envelope in which you received the exam and submit it to the proctor.

Exam continues on next page ...

Common notations:

- $\|\cdot\|_\infty$ denotes the supremum norm.
- $C[a, b]$ denotes the space of all real-valued continuous functions on $[a, b]$ equipped with the supremum norm topology.
- Ordered pair (X, ρ) denotes a metric space where X is a set and ρ is a metric on X.

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Problems:

1. Let (X, ρ) be a metric space and S be a nonempty subset of X. For $x \in X$, define the distance to S as $\text{dist}(x, S) = \inf\{\rho(x, t) : t \in S\}$.

   a. (4 pts) Show that $f(x) = \text{dist}(x, S)$ is a continuous function from X to $\mathbb{R}$ equipped with the absolute difference (Euclidean) metric.

   b. (6 pts) Show that the closure $\overline{S}$ of S satisfies $\overline{S} = \{x \in X : \text{dist}(x, S) = 0\}$.

2. For real numbers $a \leq b$ and $K \geq 0$, let $\mathcal{M}_K$ denote the set of of all functions in $C[a, b]$ which are Lipschitz continuous with constant K, i.e., $|f(x) - f(y)| \leq K|x - y|$.

   a. (5 pts) Show that for any given real number $K \geq 0$, the set $\mathcal{M}_K$ is closed in $C[a, b]$.

   b. (5 pts) Consider the set $\mathcal{M} = \bigcup_K \mathcal{M}_K$, where the union is taken over all real $K \geq 0$. Prove or disprove: $\mathcal{M}$ is closed in $C[a, b]$.

3. (10 pts) Consider the sequence of functions

$$f_n(x) = n \sin\big(\sin(n) \sin(x/n)\big), \qquad n = 1, 2, \ldots.$$

   Show that f_n are equicontinuous on $[0, 1]$. Does the sequence f_n have a uniformly converging subsequence in $C[0, 1]$?

   Hint: Prove inequality $|\sin x| \leq |x|$ by considering $\sin x = \int\limits_0^x \cos s \, ds$ and $x = \int\limits_0^x ds$.

4. Recall that a real function φ is convex on an interval $\mathcal{I}$ in the domain of φ if

$$\varphi(\lambda x + (1 - \lambda)y) \;\leq\; \lambda\varphi(x) + (1 - \lambda)\varphi(y)$$

   for all $\lambda \in [0, 1]$ and all real $x, y \in \mathcal{I}$.

   a. (6 pts) Suppose φ is convex on $\mathbb{R}$ and $\varphi(0) = 0$. Consider sets

$$\mathcal{D}_+ = \{a \in \mathbb{R} : \exists x \in (0, \infty) \text{ such that } ax = \varphi(x)\}$$

   and

$$\mathcal{D}_- = \{a \in \mathbb{R} : \exists x \in (-\infty, 0) \text{ such that } ax = \varphi(x)\}.$$

   Show that $\sup \mathcal{D}_- \leq \inf \mathcal{D}_+$. Hint: Argue by contradiction.

   b. (4 pts) Suppose φ is convex on $\mathbb{R}$. Prove that for every $\rho \in \mathbb{R}$, there exists a linear function $\ell(x) = ax + b$ satisfying $\ell(x) \leq \varphi(x)$ for all $x \in \mathbb{R}$ and $\ell(\rho) = \varphi(\rho)$. Hint: Use part (a) for the case $\rho = 0 = \varphi(\rho)$ and then use $\psi(x) = \varphi(x + \rho) - \varphi(\rho)$.