OSU Department of Mathematics
Qualifying Examination
Fall 2024

Linear Algebra

Instructions:

- Do **any three** of the four problems.
- Use separate sheets of paper for each problem. Clearly indicate the problem and page number (if several pages are used for a solution) on the top of the page.
- Your solutions should contain all mathematical details. Please write them up as clearly as possible.
- Explicitly state any standard theorems, including hypotheses, that are necessary to justify your reasoning.
- You have **four** hours to complete this examination.
- On problems with multiple parts, individual parts may be weighted differently in grading.
- When you are done with the examination:
  1. Use the problem selection sheet to indicate your identification number and the three problems which you wish to be graded.
  2. Arrange your solutions according to the problem order with the problem selection sheet on top and any scratch-work on the bottom.
  3. Submit the exam: place your solutions together with the selection sheet and scratch paper, in the order arranged as above, into the envelope in which you received the exam and submit it to the proctor.

Exam continues on next page ...

Common notations:

- $\mathbb{R}^{m\times n}$ denotes the space of real $m \times n$ matrices with m rows and n columns.
- $\mathbb{C}^{m\times n}$ denotes the space of complex $m \times n$ matrices with m rows and n columns.
- Respectively, $\mathbb{C}^{n\times 1}$ is the space of n-dimensional column vectors.
- For $A \in \mathbb{C}^{m\times n}$, matrix $A^T \in \mathbb{C}^{n\times m}$ denotes its transpose.
- For $A \in \mathbb{C}^{m\times n}$, matrix $A^* \in \mathbb{C}^{n\times m}$ denotes its conjugate transpose.
- $N(M)$ denotes the *null space* of matrix M.

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Problems:

1. **a.** (2 pts) Show that for $y \in \mathbb{C}^{n\times 1}$, $y^*y = 0$ if and only if $y = 0$ (zero vector).

   b. (3 pts) For $A \in \mathbb{C}^{m\times n}$ and $x \in \mathbb{C}^{n\times 1}$, show that $Ax = 0$ if and only if $A^*Ax = 0$.

   c. (3 pts) For $A \in \mathbb{C}^{m\times n}$, show that $\operatorname{rank}(A) = \operatorname{rank}(A^*A)$.

   d. (2 pts) For $A \in \mathbb{C}^{m\times n}$, show that $\operatorname{rank}(A) = \operatorname{rank}(A^*)$.

2. A real symmetric matrix $A \in \mathbb{R}^{n\times n}$ is *positive definite* if $x^TAx > 0$ for every nonzero vector $x \in \mathbb{R}^{n\times 1}$. Let Σ_n be the set of all real symmetric positive definite $n\times n$ matrices.

   a. (2 pts) Show that if $A \in \Sigma_n$ and $B \in \Sigma_n$, then $A + B \in \Sigma_n$.

   b. (3 pts) For $A \in \mathbb{R}^{n\times n}$, show that $A \in \Sigma_n$ if and only if

   $$M^TAM \in \Sigma_m \qquad \forall m \in \{1, \dots, n\} \text{ and all } M \in \mathbb{R}^{n\times m} \text{ with } N(M) = \{0\}.$$

   c. (3 pts) For $A \in \mathbb{R}^{n\times n}$, show that $A \in \Sigma_n$ if and only if there exists an invertible real matrix $Q \in \mathbb{R}^{n\times n}$ such that $A = QQ^T$.

   d. (2 pts) Suppose $A \in \Sigma_n$. Show that for any positive integer k, there exists a matrix $R \in \Sigma_n$ such that $A = R^k$.

3. Let V be a finite dimensional complex vector space. Recall that a linear operator $T : V \to V$ is said to be *nilpotent* if $T^m = 0$ for some integer $m \geq 1$.

   a. (2 pts) Suppose that $T : V \to V$ is a linear operator. Prove that if T is nilpotent, then $T^n = 0$, where $n = \dim V$.

   b. (2 pts) Prove that the only linear operator $T : V \to V$ that is both diagonalizable and nilpotent is the zero operator.

   c. (6 pts) Suppose that $T : V \to V$ is a linear operator. Prove that there exist linear operators $D : V \to V$ and $N : V \to V$ such that D is diagonalizable, N is nilpotent, $T = D + N$, and both D and N commute with T; in other words $DT = TD$ and $NT = TN$.

4. Let A be an orthogonal 3×3 real matrix.

   a. (2 pts) Prove that $\det(A) = \pm 1$.

   b. (3 pts) Let $f(x)$ be the characteristic polynomial of A. Prove that if $\det(A) = 1$ then $f(1) = 0$.

   c. (5 pts) Consider the subspace of $\mathbb{R}^3$ defined by

   $$\operatorname{Fix}(A) = \{\mathbf{x} \in \mathbb{R}^3 \mid A\mathbf{x} = \mathbf{x}\}.$$

   Prove that if $\det(A) = 1$ and A is not the identity matrix, then $\dim \operatorname{Fix}(A) = 1$.